

Using Shannon’s entropy to quantify preference heterogeneity in large populations with complete preorders

S. Prieto-Herráez^{1†}, J.M. Cascón^{2†}, R. de Andrés Calle^{3*}

¹*PRESAD Research Group, Universidad de Salamanca, E37007 Salamanca, Spain.

²Department of Economics and Economic History, Institute on Fundamental Physics and Mathematics, Universidad de Salamanca, E37007 Salamanca, Spain.

³BORDA and PRESAD Research Groups, Research Excellence Unit GECOS and Multidisciplinary Institute of Enterprise (IME) University of Salamanca, E37007 Salamanca, Spain.

*Corresponding author(s). E-mail(s): rocioac@usal.es;

Contributing authors: silvi_ph@usal.es; casbar@usal.es;

[†]These authors contributed equally to this work.

Abstract

Understanding how much a group of individuals disagrees in their preferences is a fundamental problem in collective decision-making. When the number of participants and available options is large, measuring this disagreement rigorously and efficiently becomes particularly demanding. The present study addresses this challenge by introducing a formally grounded measure of preference heterogeneity for large-scale group decision-making settings, in which individual preferences are expressed as complete rankings over a set of alternatives, allowing for ties. The proposed measure is derived from Shannon’s entropy and quantifies the degree of diversity in the distribution of individual preferences across a group. A value of zero indicates full agreement, whereas higher values reflect greater dispersion of opinions. The measure is shown to satisfy a set of desirable normative properties drawn from Social Choice Theory, including anonymity, neutrality, and conditions governing how heterogeneity changes when the group or the set of alternatives is modified. An axiomatic characterisation establishes that the proposed measure is the unique index satisfying these properties jointly. The computational simplicity of the measure makes it applicable to very large populations

without loss of tractability. Its practical performance is evaluated through ten real-world ranked-choice elections from a publicly available preference database, encompassing electorates of up to nearly three hundred thousand voters and up to eleven candidates. Simulation experiments conducted under diverse probabilistic scenarios further confirm the robustness and sensitivity of the measure to variations in preference dispersion.

Keywords: Information-theoretic heterogeneity, Complete preorders, Preference profiles, Uncertainty quantification, Behavioural diversity

1 Introduction

Preferences play a fundamental role in decision-making, shaping choices at both individual and collective levels (see, for instance, [1]; [2]; [3]; [4]). The study of decision-making is inherently multifaceted, encompassing a wide range of perspectives aimed at understanding the complexity of human behaviour. While some research focuses on the cognitive and contextual factors that influence individual preferences, other approaches address collective decision-making, where the aggregation of diverse individual opinions introduces additional challenges. In particular, group decision-making often faces substantial difficulties in achieving consensual outcomes because of preference heterogeneity (see [2], [5], [6], [7], and [8], among others).

Measuring agreement in group decision-making has been extensively discussed in fields such as *Social Choice Theory* and *Decision-Making Theory*. The notion of “*consensus*” has been interpreted in multiple ways (see [9] and [10]). In the present work, consensus is understood as a static evaluation of the degree of agreement within a preference profile, without requiring individuals to modify their expressed positions¹. The aim here is to provide a global measure of cohesion based solely on the structure of the observed preferences.

Achieving this goal requires formalising how preferences are represented and compared. In many applications, preferences over a finite set of alternatives can be encoded as rank vectors. However, analysing such preference data becomes increasingly complex as the number of individuals and alternatives grows. A major challenge lies in measuring preference heterogeneity without compromising the interpretability of the underlying decision behaviour. This issue becomes particularly salient in large-scale settings, where datasets often involve thousands of agents and numerous alternatives.

To address this challenge, the present study introduces an entropy-based approach to quantify preference heterogeneity in group decision-making. The proposed use of Shannon’s entropy is rooted in its well-established role in Information Theory as a measure of uncertainty, but also draws conceptual inspiration from Statistical Mechanics, where entropy reflects the degree of disorder in complex systems. This analogy is particularly suitable for the analysis of large-scale preference structures.

¹A related line of research investigates consensus-reaching processes, where individuals iteratively adjust their preferences, often under the influence of a moderator [11].

Entropy has been used in previous studies to capture disagreement among preferences (see, e.g., [12], [13], [14] and [15]), but typically under simplified frameworks based on Likert-type scales, linear orders, fuzzy relations, or cardinal utilities. These approaches do not consider the more general and demanding case of complete preorders, which accommodate both strict preferences and indifference, and raise additional theoretical and computational challenges.

Against this backdrop, the first objective of this study is to develop an entropy-based approach for measuring preference heterogeneity, specifically in contexts where individual preferences are modelled as complete preorders over a finite set of alternatives. This class of preferences is more expressive than the simplified structures used in previous work and covers a broader range of decision behaviours.

Moreover, unlike previous contributions that apply entropy heuristically or in a purely descriptive manner, the present paper introduces a formally defined heterogeneity measure based on Shannon’s entropy. This formulation is supported by a rigorous mathematical foundation and satisfies several desirable properties, including continuity, replication invariance, convergence to unanimity, and alternative invariance. These properties align with key normative principles from Social Choice Theory, including unanimity, neutrality, and anonymity.

From an information-theoretic perspective, Shannon’s entropy offers a principled and computationally tractable means of quantifying the uncertainty inherent in a probability distribution and, by extension, the degree of dispersion in a group’s expressed preferences. A detailed justification for the choice of Shannon’s entropy over alternative measures is provided in Section 2.4.

The second aim of this work is to analyse the main mathematical properties of the proposed measure, including continuity, boundedness and grouping, and to provide an axiomatic characterisation that uniquely identifies it under this set of normative conditions.

The third objective is to evaluate the practical relevance of the measure through empirical analysis, comparison with alternative indicators, and simulation experiments. The measure is applied to ten real ranked-choice elections retrieved from the *PrefLib* database, covering different electorate sizes, numbers of alternatives, observed preorder supports, and preference distributions. A complementary comparison with selected consensus- and distance-based indicators is then conducted for the same profiles, providing insight into commonalities and differences between the proposed information-theoretic index and existing measures of consensus or dissent. Finally, simulation experiments based on different probabilistic models are used to assess the response of the measure to controlled changes in concentration, multimodality, skewness, and the number of distinct preorder types. Taken together, these analyses support the interpretability, scalability, and practical usefulness of the proposed measure in large-scale group decision-making settings.

The structure of the paper is as follows. Section 2 presents a review of the main approaches for measuring preference heterogeneity and discusses the rationale for adopting Shannon entropy over alternative measures. Section 3 introduces

the required preliminary concepts and assumptions. Section 4 defines the proposed entropy-based measure and examines its properties, including its axiomatic characterisation. Section 5 provides empirical illustrations and simulation results. Section 6 offers concluding remarks and outlines potential directions for future research. Finally, the paper includes three appendices. Appendix A collects the notation used throughout the paper, together with a summary of the main properties of the proposed measure and their normative role. Appendix B contains the proofs of the properties established in Subsection 4.1. Appendix C presents the proof of the axiomatic characterisation of the measure.

2 Heterogeneity preference measurements in the literature

A substantial body of literature has addressed the measurement of preference heterogeneity, also referred to as consensus measurement (see [16] for a review of definitions), drawing on methodologies ranging from simple dispersion indices to axiomatic frameworks and information-theoretic approaches. This section surveys the main contributions, organised around three interconnected areas, and concludes with a discussion of the rationale for adopting Shannon’s entropy in the present work.

The first area, discussed in Section 2.1, explores research from Social Choice Theory, which characterises heterogeneity using axiomatic notions of agreement and distance-based metrics applied to rankings and preorders. The second area, covered in Section 2.2, focuses on Decision-Making Theory, particularly consensus frameworks that employ fuzzy and linguistic representations. The third area, presented in Section 2.3, examines entropy-based approaches, emphasising how measures derived from Shannon’s work quantify disagreement within Likert-scale and fuzzy systems. Table 1 provides an integrated summary of these contributions. The section concludes with Section 2.4, which explains the rationale for specifically employing Shannon’s entropy in this research.

2.1 Approaches from Social Choice Theory

From the perspective of *Social Choice Theory*, Bosch’s PhD thesis introduced the first thorough examination and discussion of consensus measurement from an *Arrovian* perspective [17]. McMorris and Powers characterised consensus rules defined on hierarchies [18], and García-Lapresta and Pérez-Roman examined consensus measurement based on distances, considering complete preorders on alternatives [19]. Subsequently, a family of linear and additive consensus measures was characterised by Alcalde-Unzu and Vorsatz, and novel approaches to measuring preference similarity in a group were proposed (see [20] and [21], respectively).

Alcantud, de Andrés Calle and Cascón conducted research on a particular type of consensus measure, the *referenced consensus measure*. This measure enables the construction of a numerical social evaluation using only ordinal individual data (for details, see [22]). Furthermore, in [23], the authors made valuable contributions to the analysis of the consensus measure mentioned above. Their analysis specifically centred

on two notable scenarios: the Borda rule and the Copeland rule, both evaluated using a Kemeny-type measure.

From an alternative standpoint, in certain settings, individuals in a group may categorise a set of choices as either acceptable or non-acceptable, agree or disagree, etc., resulting in a binary preference structure. Based on this idea, Alcantud, de Andrés Calle, and Cascón introduced the notion of an “approval consensus measure” and provided axiomatic characterisations for two general types of these measures [24]. Erdamar et al. proposed a distance-based approach to measuring consensus, taking into account approval information and rankings of alternatives [25].

Finally, González-Arteaga, Alcantud and de Andrés Calle pioneered the application of the Mahalanobis distance to the evaluation of heterogeneity in cardinal evaluations and complete preorders (see [26] and [27], respectively).

2.2 Approaches from Decision-Making Theory

Within Decision-Making Theory, the foundational contribution of Kacprzyk and Fedrizzi [28] introduced the concept of the “degree of consensus” under fuzzy preference relations, establishing a framework that has since generated a substantial family of consensus measures [29–34]. A parallel research strand has examined heterogeneity measurement when expert opinions are expressed through linguistic assessments, employing multigranular, hesitant, or interval-valued representations [35–41].

2.3 Approaches based on Entropy Concepts

Entropy has emerged as a powerful and widely used tool for quantifying uncertainty and diversity in decision-making processes. One of the earliest applications can be found in the work of [42], who proposed Social Entropy Theory to study disorder and structure in social systems, including group judgments. Although this contribution is conceptual and sociological, it helped set the stage for quantitative entropy-based approaches to disagreement.

A major advance came with the work of [12], who developed a consensus and dissension metric explicitly based on Shannon entropy, designed to quantify the degree of ordinal dispersion in group responses on Likert-type scales. This proposal interprets consensus as a function of how concentrated or dispersed the responses are around the mean of the distribution, making it one of the first direct and interpretable entropy-based consensus measures for ordinal data.

Later, [43] proposed a refined measure of disagreement based on conditional probabilities, formulating an entropy-based index that assesses consensus within groups using Likert-scale data. This approach extends the idea of entropy to account for conditional variability in responses, offering a probabilistic interpretation of group consensus.

In the context of fuzzy preference relations, [14] introduced a consensus index grounded in Theil’s entropy. This approach evaluates the level of agreement among group members based on the randomness of fuzzy opinions. A comparative analysis with traditional distance-based methods reveals that the entropy-based index behaves similarly when applied to fuzzy settings, showing its usefulness for capturing consensus

under uncertainty [44]. However, its application is limited to fuzzy information and does not extend to ordinal or complete preference structures.

Additional applications have emerged in multi-criteria decision-making contexts. For instance, [45] introduced a consensus model that combines majority opinions with entropy support to identify representative group decisions. Similarly, [46] proposed a generalised entropy-based weighting method for group decision-making, which can be used to assess the importance of decision criteria or alternatives based on their informational contribution to the overall system.

These contributions represent important steps forward in the application of entropy to preference analysis. However, most existing models are context-dependent and tailored to specific data structures, such as Likert-type ratings, fuzzy relations, or weighted voting. Furthermore, although they provide intuitive interpretations, they are not typically grounded in a formal axiomatic theory of preferences. In addition, only a few of these models explicitly connect entropy measurement to a statistical-mechanics-inspired interpretation of preference distributions.

On the other hand, some extensions of the entropy concept have appeared in the literature, such as Quantum X-entropy [47, 48], which generalises von Neumann entropy for quantum systems, and f-divergence [49], which measures dissimilarity between probability or belief functions. While these approaches are relevant in their respective domains, their application to the present context does not appear straightforward, as the focus here is on adapting Shannon entropy to measure heterogeneity in ordinal or preorder-based preferences.

2.4 Why Shannon Entropy for Measuring Preference Heterogeneity?

The choice of Shannon’s entropy in this contribution is justified not merely by computational convenience, but also by a clear normative and operational rationale. Starting from the observed profile, an empirical distribution over the complete preorder types present in the group is induced; Shannon’s entropy then provides a canonical scalar summary of that distribution, increasing with both the number of distinct types and the evenness of their frequencies. This choice is further supported by an *aggregation* (*coarse-graining*) principle: when preference categories are merged into a coarser category, the resulting loss of heterogeneity should decompose coherently into between-group and within-group components. Under continuity, boundedness, and this consistency requirement under category aggregation, Shannon’s entropy is uniquely singled out within the class of distribution-based indices.

The interpretation is also immediate. Entropy quantifies the uncertainty about the preorder type of a randomly selected individual. This reading is particularly suitable for ordinal data with indifferences, since it does not rely on metric embeddings or pairwise interpersonal comparisons.

This reading also connects the proposed approach with the notion of *social homogeneity* studied in voting theory [50, 51]. In that literature, group coherence is gauged by how likely two agents chosen at random are to share the same preference, whereas the entropy-based perspective adopted here gauges the uncertainty about the

Table 1: Summary of heterogeneity measures in the literature.

Approach: Social Choice Theory		
Reference	Type of Measure	Type of Information
[17]	Conceptual reasoning	Linear orders
[18]	Consensus rules	Linear orders
[19]	Distance-based consensus	Complete orders
[20]	Linear and additive measures	Linear orders
[22]	Referenced consensus measure	Complete orders
[23]	Kemeny-type distance	Complete orders
[24]	Approval consensus measure	Binary
[25]	Distance-based consensus	Binary + Ordinal
[21]	Similarity index	Linear orders
[26]	Mahalanobis distance	Cardinal
[27]	Mahalanobis distance	Complete orders
Approach: Decision-Making Theory		
Reference	Type of Measure	Type of Information
[28]	Degree of consensus	Fuzzy preferences
[35]	Consensus index	Linguistic
[29]	Soft Consensus	Fuzzy preferences
[36]	Linguistic consensus index	Multigranular
[37]	Consensus measure	Linguistic
[38]	Agreement index	Unbalanced
[30]	Aggregation functions	Fuzzy
[31]	Consensus index	Interval-valued
[39]	Distance-based consensus	Hesitant
[40]	Consensus index	Incomplete
[32]	Correlation degree	Fuzzy preferences
[41]	Multicriteria	Multigranular
[33]	Similarity-based index	Fuzzy
Approach: Entropy-Based Methods		
Reference	Type of Measure	Type of Information
[42]	Conceptual entropy theory	General / sociological
[12]	Modified Shannon entropy	Likert scale
[45]	Entropy-based aggregation method	Cardinal
[44]	Uncertainty measure	Linear orders
[43]	Entropy-based index	Likert scale
[46]	Generalised entropy weights	Ordinal or cardinal
[14]	Theil-type entropy index	Fuzzy preferences

preference of a randomly selected agent. Both viewpoints summarise the concentration or dispersion of a collective profile and coincide at the extremes: full coherence under unanimity and maximal dispersion when all preference types are equally represented. They differ, however, in how intermediate dispersion is weighted. The proposed measure may thus be regarded as an information-theoretic counterpart to classical social-homogeneity indicators, adapted to the complete-preorder setting. A formal statement of this correspondence is deferred to Section 4, once the required notation has been introduced.

The computational and empirical advantages of the proposed measure, including its linear complexity with respect to the number of observed preorder types and its continuity-based robustness to small perturbations in the preference distribution, are formally established in Subsection 4.1 and demonstrated empirically in Section 5.

Table 2 synthesises the comparative analysis developed across Subsections 2.1–2.3, contrasting the three main methodological strands with respect to their structural domain, computational scalability, axiomatic grounding, principal limitation, and data requirements. The proposed entropy-based measure, presented in the final row, addresses the gaps identified in prior work by operating on complete preorders, thereby accommodating indifference, and by providing a formally axiomatised, computationally efficient index applicable to large-scale decision-making settings.

Table 2: Methodological positioning of the proposed entropy-based measure.

Approach		Preference structure	Main strength	Main limitation
Social Theory	Choice	Linear orders, complete preorders, and binary evaluations	Well-established distance- and similarity-based metrics with strong normative foundations	Many measures rely on pairwise comparisons or distance computations, which may become computationally demanding in large-scale settings.
Decision-Making Theory		Fuzzy preferences, linguistic assessments, interval-valued or incomplete information	Flexible modelling of uncertainty, vagueness, and expert assessments	Measures are often context-dependent and may depend on the chosen linguistic, fuzzy, or aggregation framework.
Entropy-based methods		Likert scales, fuzzy preferences, cardinal or ordinal evaluations	Natural interpretation in terms of uncertainty, dispersion, or informational diversity	Most existing proposals are designed for specific data structures and are not formulated for complete preorders.
Proposed measure		Complete preorders	Distribution-based, scalable, and axiomatically characterised entropy measure	Currently formulated for complete preorders; extensions to partial, truncated, or noisy preferences remain a direction for future research.

3 Notation and starting hypothesis

This section introduces the basic definitions and outlines the problem formulation.

Let $\mathbf{N} = \{1, \dots, n\}$ be a group of agents, namely the *agent set*, with $n = |\mathbf{N}| > 2$, and let $\mathbf{X} = \{x_1, \dots, x_k\}$ be a finite set of alternatives, with $k = |\mathbf{X}| \geq 2$.

It is assumed throughout that agents express their preferences over alternatives by means of complete preorders, that is, reflexive, transitive, and *complete* (total) binary relations on $\mathbf{X}$, so that any two alternatives are always comparable, with indifference corresponding to the case in which each is regarded as at least as good as the other. This assumption allows the model to reflect the structure of decision-making situations in which indifference between alternatives may occur.

The set of all complete preorders on $\mathbf{X}$ is denoted by $\mathbf{W}(\mathbf{X})$, and $\phi(k)$ denotes its cardinality², that is, $\phi(k) = |\mathbf{W}(\mathbf{X})|$. Moreover, an enumeration, namely a complete ordered listing, of the elements in $\mathbf{W}(\mathbf{X})$ is assumed.

Let $\mathcal{R}_i \in \mathbf{W}(\mathbf{X})$ be the *preference* of agent i on the set of alternatives $\mathbf{X}$. The notation $x_s \mathcal{R}_i x_t$ means that alternative x_s is at least as good as x_t for agent i .

Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{W}(\mathbf{X}) \times \dots \times \mathbf{W}(\mathbf{X}) = \mathbf{W}(\mathbf{X})^n$ be a *preference profile* of the group of agents $\mathbf{N}$ on the set of alternatives $\mathbf{X}$. Let $\mathbf{P}(\mathbf{X})$ denote the set of all preference profiles, that is, $\mathbf{P}(\mathbf{X}) = \bigcup_{n \geq 2} \mathbf{W}(\mathbf{X})^n$. The preference profile $\mathcal{P}$ is unanimous with respect to $\mathcal{R}$ if $\mathcal{P} = (\mathcal{R}, \dots, \mathcal{R})$, i.e., the preferences on $\mathbf{X}$ are the same for all agents.

Any permutation π of the alternatives $\{x_1, \dots, x_k\}$ determines a permutation of every preference $\mathcal{R}_i \in \mathbf{W}(\mathbf{X})$ via $x_s \pi \mathcal{R}_i x_t \iff x_{\pi(s)} \mathcal{R}_i x_{\pi(t)}$ for all $s, t \in \{1, \dots, k\}$ and $i \in \{1, \dots, n\}$. Consequently, to each $\mathcal{P}$ and π , there is associated a preference profile $\pi \mathcal{P} = (\pi \mathcal{R}_1, \dots, \pi \mathcal{R}_n)$. Likewise, any permutation σ of the agents $\{1, \dots, n\}$ determines a permutation of $\mathcal{P}$ by $\mathcal{P}^\sigma = (\mathcal{R}_{\sigma(1)}, \dots, \mathcal{R}_{\sigma(n)})$.

For each preference profile $\mathcal{P} \in \mathbf{W}(\mathbf{X})^n$, its restriction to a *subprofile* of the agents in $I \subseteq \mathbf{N}$ is denoted by $\mathcal{P}^I$ and is obtained by selecting precisely those preferences of $\mathcal{P}$ that are associated with the agents³ in I . For brevity, $\mathcal{P}^i$ is written instead of $\mathcal{P}^I$ when $I = \{i\}$. Any partition $\{I_1, \dots, I_s\}$ of $\mathbf{N}$ generates a decomposition of $\mathcal{P}$ into preference subprofiles $\mathcal{P}^{I_1}, \dots, \mathcal{P}^{I_s}$ such that $\mathcal{P}^{I_1 \cup I_2 \cup \dots \cup I_s} = \mathcal{P}$.

Given a subset $\mathbf{X}_A$ of $\mathbf{X}$, $\mathcal{R}|_{\mathbf{X}_A}$ denotes the restriction of $\mathcal{R}$ to $\mathbf{X}_A$ and $\mathcal{P}|_{\mathbf{X}_A}$ denotes the restriction of $\mathcal{P}$ to $\mathbf{X}_A$.

Suppose that a set of q new agents is incorporated into $\mathbf{N}$ in the following manner, $\bar{\mathbf{N}}^q = \{1, \dots, n, n+1, \dots, n+q\}$. The set $\bar{\mathbf{N}}^q$ is called the *extended agent set*. An *extension* of a preference profile $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ is a new profile, $\bar{\mathcal{P}}^q$, such that it includes the preferences of the q agents added to $\mathbf{N}$, i.e., $\bar{\mathcal{P}}^q = (\mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{R}_{n+1}, \dots, \mathcal{R}_{n+q}) \in \mathbf{P}(\mathbf{X})$. For simplicity, $\bar{\mathcal{P}}$ is written instead of $\bar{\mathcal{P}}^1$.

Suppose that q new alternatives are added to $\mathbf{X}$, such that $\tilde{\mathbf{X}}^q = \{x_1, \dots, x_k, x_{k+1}, \dots, x_{k+q}\}$ denotes the *expanded set of alternatives*. The set of all complete preorders on $\tilde{\mathbf{X}}^q$ is denoted by $\mathbf{W}(\tilde{\mathbf{X}}^q)$. $\tilde{\mathcal{R}}_i \in \mathbf{W}(\tilde{\mathbf{X}}^q)$ is called the *expanded preference* of agent i on $\tilde{\mathbf{X}}^q$. An *expanded preference profile* $\tilde{\mathcal{P}}^q = (\tilde{\mathcal{R}}_1, \dots, \tilde{\mathcal{R}}_n) \in \mathbf{W}(\tilde{\mathbf{X}}^q)^n$ is a new preference profile such that each $\tilde{\mathcal{R}}_i$ is the expanded preference of agent i on $\tilde{\mathbf{X}}^q$. Let $\mathbf{P}(\tilde{\mathbf{X}}^q)$ denote the set of all expanded preference profiles.

Finally, a replication of a preference profile $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{W}(\mathbf{X})^n$ of the group of agents $\mathbf{N}$ on $\mathbf{X}$ is the profile $\mathcal{P} \uplus \mathcal{P} \in \mathbf{W}(\mathbf{X})^{2n}$ obtained by duplicating each

²Note that $\phi(k) = \sum_{j=0}^k j! \cdot S(k, j)$ where $S(k, j)$ is the Stirling number of the second kind (see [52] for details).

³Here I is a subset of $\mathbf{N}$ that contains, in the same order, elements of $\mathbf{N}$.

individual preference. Thus,

$$\mathcal{P} \uplus \mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{R}_1, \dots, \mathcal{R}_n).$$

Equivalently, for each $i = 1, \dots, n$, the preference occupying position $n + i$ in the replicated profile is identical to the preference $\mathcal{R}_i$ occupying position i in the original profile.

For clarity, the notation employed throughout the paper is summarised in Appendix A, Table A1.

The next definition formalises the distribution associated with a profile $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ under a fixed enumeration of $\mathbf{W}(\mathbf{X})$.

Definition 1 Assume an enumeration of $\mathbf{W}(\mathbf{X})$. Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile. The *distribution* of the profile $\mathcal{P}$ is a discrete probability distribution, $p(\mathcal{P}) \equiv (p_i)_{i=1}^{\phi(k)}$, whose i -th component is the proportion, or relative frequency, of agents in $\mathcal{P}$ whose preference coincides with the i -th preorder of $\mathbf{W}(\mathbf{X})$, that is,

$$p : \mathbf{P}(\mathbf{X}) \longrightarrow [0, 1]^{\phi(k)}$$

$$\mathcal{P} \longrightarrow p(\mathcal{P}) = (p_1, \dots, p_{\phi(k)}) = (p_i)_{i=1}^{\phi(k)},$$

with

$$p_i = \frac{\#\{\mathcal{R}_j \in \mathcal{P} : \mathcal{R}_i \equiv \mathcal{R}_j\}}{n}. \quad (1)$$

When there is no ambiguity, the dependence of the distribution components on the profile will be omitted, i.e., $p_i \equiv p_i(\mathcal{P})$.

Remark 1 Since $p_i \geq 0$ for $i = 1, \dots, \phi(k)$ and $\sum_{i=1}^{\phi(k)} p_i = 1$, it follows that $p(\mathcal{P})$ is a probability mass function.

Remark 2 The discrete probability distribution $p(\mathcal{P})$ may be interpreted as an experimental probability, or relative frequency, of the preorders in $\mathcal{P}$, since it is obtained by dividing the number of agents holding each preorder by the total number of agents in $\mathcal{P}$. Two readings of this distribution must be distinguished. Under the *descriptive* reading adopted throughout this paper, the observed profile constitutes a complete enumeration, effectively a census, of the group under study; $p(\mathcal{P})$ is then the exact relative-frequency distribution of the preorder types within that group, and any quantity computed from it describes the observed group, with no inferential claim about an unobserved population. This reading is appropriate because group decision-making problems are not standard statistical experiments, and there may be no possibility of repeating them an indefinite number of times; accordingly, the approach adopted here is grounded in the frequency interpretation of probability, one of the most widely used interpretations of the concept (further details can be found in [53], [54], [55] and [56], among others).

Under an *inferential* reading, by contrast, the observed profile would be regarded as a finite sample drawn from a larger population, and $p(\mathcal{P})$ as the plug-in, or relative-frequency,

estimator of the population type-probabilities. In that case, the estimator is known to be negatively biased in finite samples, and any generalisation beyond the observed agents would require sampling-based uncertainty quantification, for instance through a Miller–Madow bias correction or resampling (bootstrap) confidence intervals. The continuity property established in Subsection 4.1 ensures that such sampling perturbations induce only small variations in the index, which supports the stability of comparisons across samples; a formal inferential treatment lies beyond the present scope and is identified as a direction for future research.

The following example is presented to illustrate the concept of the *discrete probability distribution* of $\mathcal{P}$.

Example 1 Let $\mathbf{N} = \{1, \dots, 5\}$ be a set of five agents who provide their preferences over the set of three alternatives $\mathbf{X} = \{x_1, x_2, x_3\}$ by means of the following preference profile:

$$\mathcal{P} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5) = \begin{pmatrix} x_1 \succ x_2 \succ x_3 \\ x_3 \succ x_2 \sim x_1 \\ x_1 \succ x_2 \succ x_3 \\ x_3 \sim x_2 \succ x_1 \\ x_3 \succ x_1 \succ x_2 \end{pmatrix} \in \mathbf{W}(\mathbf{X})^5.$$

To determine the discrete probability distribution associated with $\mathcal{P}$, an enumeration of the thirteen complete preorders on the three alternatives must first be fixed. Table 3 provides the enumeration adopted in this example. Accordingly, the discrete probability distribution of $\mathcal{P}$ is

$$p(\mathcal{P}) = \left(0, 0, 0, \frac{2}{5}, 0, 0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0\right).$$

The methodological approach developed in this contribution is independent of the particular enumeration of the preorders. An enumeration of $\mathbf{W}(\mathbf{X})$ is required only when the discrete probability distribution is written in vector form. Since the entropy-based heterogeneity measure depends only on the collection of strictly positive frequencies and not on their positions in the enumeration, its value is unaffected by this choice.

Table 3: Enumeration of all complete preorders in $\mathbf{W}(\mathbf{X})$, for $\mathbf{X} = \{x_1, x_2, x_3\}$.

<i>i</i> -th : $\mathcal{R}$		
1: $x_1 \sim x_2 \sim x_3$	6: $x_1 \succ x_3 \succ x_2$	10: $x_3 \succ x_1 \succ x_2$
2: $x_1 \sim x_2 \succ x_3$	7: $x_2 \succ x_1 \sim x_3$	11: $x_2 \sim x_3 \succ x_1$
3: $x_1 \succ x_2 \sim x_3$	8: $x_2 \succ x_1 \succ x_3$	12: $x_2 \succ x_3 \succ x_1$
4: $x_1 \succ x_2 \succ x_3$	9: $x_3 \succ x_1 \sim x_2$	13: $x_3 \succ x_2 \succ x_1$
5: $x_1 \sim x_3 \succ x_2$		

4 Heterogeneity preference measure: an approach based on Shannon entropy

From the perspective of the representational theory of measurement, preference structures can be represented in various ways (see, e.g., [57], [58], [59], [60] and [61]). Depending on the representation adopted, different econometric and statistical studies have examined the measurement of preference heterogeneity. To the best of current knowledge, however, a general theoretical definition of a preference heterogeneity measure has not yet been established. Related empirical and econometric treatments of preference heterogeneity can be found, for instance, in [62], [63], [64], [65] and [66], among others.

This section develops a theoretical framework for measuring preference heterogeneity, with particular emphasis on the *entropy-based heterogeneity preference measure*. The contribution lies in the formal definition of the object to be measured, the introduction of an entropy-based specification, and the analysis of the normative properties satisfied by this specification.

Definition 2 Let $P = (R_1, \dots, R_n) \in \mathcal{P}(X)$ be a preference profile. A heterogeneity preference measure on $\mathcal{P}(X)$ is a mapping

$$\delta : \mathcal{P}(X) \longrightarrow [0, \infty)$$

that assigns a non-negative value to every preference profile and satisfies

$$\delta(\mathcal{P}) = 0 \iff R_1 = \dots = R_n.$$

That is, $\delta(\mathcal{P}) = 0$ if and only if the preference profile is unanimous. This condition is referred to as minimum preference heterogeneity.

The following additional requirements will be considered.

1. *Preference neutrality*. For every permutation π of the alternatives,

$$\delta(\pi\mathcal{P}) = \delta(\mathcal{P}).$$

Thus, the heterogeneity value is invariant under relabelling of the alternatives.

2. *Preference anonymity*. For every permutation σ of the agents,

$$\delta(\mathcal{P}^\sigma) = \delta(\mathcal{P}).$$

Thus, the heterogeneity value is invariant under relabelling of the agents.

A specific preference heterogeneity measure is now introduced. The proposed measure takes as input the discrete probability distribution induced by a preference profile over the set of complete preorders and assigns to it a single non-negative value. This value is equal to zero under complete preference homogeneity and is strictly positive whenever more than one preorder type is observed in the profile.

Definition 3 Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile. The *entropy-based heterogeneity preference measure* on the preference profile $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ is the mapping $\delta_H : \mathbf{P}(\mathbf{X}) \rightarrow [0, \infty)$ given by

$$\delta_H(\mathcal{P}) = - \sum_{\substack{i=1, \dots, \phi(k) \\ p_i > 0}} p_i \log p_i, \quad (2)$$

where $(p_i)_{i=1}^{\phi(k)}$ denotes the discrete probability distribution of the profile $\mathcal{P}$.

Remark 3 The condition $p_i > 0$ is imposed because the logarithm function is not defined at zero. Nevertheless, since $\lim_{x \rightarrow 0^+} x \log x = 0$, the previous definition can be extended by continuity to $p_i = 0$. From now on, the condition $p_i > 0$ will be omitted for simplicity.

Remark 4 The entropy-based heterogeneity preference measure admits a natural reading alongside the social-homogeneity indicators used in voting theory [50, 51]. A prominent such indicator is the coincidence probability

$$\sigma(p) = \sum_i p_i^2,$$

namely the probability that two agents drawn at random from the group hold the same complete preorder, whereas

$$\delta_H(\mathcal{P}) = - \sum_i p_i \log p_i$$

quantifies the uncertainty about the preorder type of a randomly selected agent. The two quantities are aligned at the structural extremes: under unanimity, $\sigma(p) = 1$ and $\delta_H(\mathcal{P}) = 0$; under the uniform distribution over $\mathbf{W}(\mathbf{X})$, $\sigma(p) = 1/\phi(k)$ and $\delta_H(\mathcal{P}) = \log \phi(k)$. They differ in how intermediate dispersion is weighted⁴.

Remark 5 Building on the inferential reading discussed in Remark 2, when the observed profile is regarded as a finite sample, $\delta_H(\mathcal{P})$ acts as the plug-in estimator of the population entropy, which is negatively biased in finite samples; corrections such as the Miller–Madow estimator [67], or bootstrap confidence intervals, may then be applied. The bias and consistency of plug-in entropy estimators are studied in detail by Paninski [68].

In applications involving different numbers of alternatives, it is useful to work with a normalised version of the entropy-based heterogeneity preference measure, so that values are expressed on a common unit interval.

Definition 4 Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathcal{P}(\mathbf{X})$ be a preference profile over a finite set of alternatives $\mathbf{X}$, with $|\mathbf{X}| = k \geq 2$. The *relative entropy-based heterogeneity preference measure* is defined by

$$\delta_{H_R}(\mathcal{P}) = \frac{\delta_H(\mathcal{P})}{\log(\phi(k))}.$$

⁴The coincidence probability is closely related to the order-two Rényi entropy, since $-\log \sigma(p)$ coincides with $H_2(p)$, whereas the proposed measure corresponds to the limiting order-one case, the Shannon entropy $H_1(p)$. Under the recursive grouping condition used in Subsection 4.2, this limiting case is the entropy functional selected by the characterisation theorem.

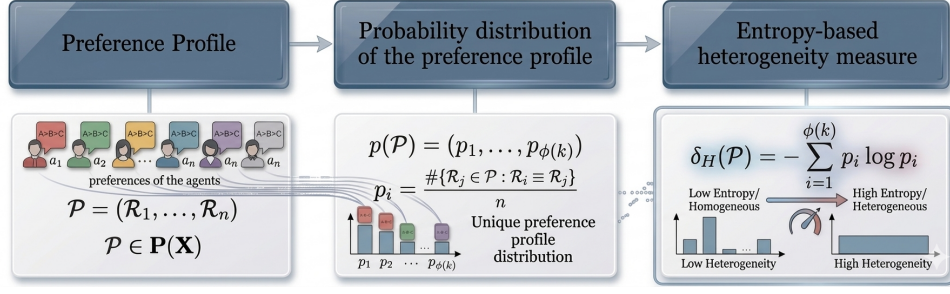


Figure 1: Sequential process from the preference profile to the entropy-based heterogeneity measure.

Since $\phi(k)$ denotes the number of complete preorders on $\mathbf{X}$, the denominator $\log(\phi(k))$ is the maximum attainable value of δ_H (see Subsection 4.1). Hence, this measure takes values in $[0, 1]$, where 0 corresponds to unanimity and 1 corresponds to the case in which all complete preorders in $\mathbf{W}(\mathbf{X})$ are equally represented.

Note that the definition of the entropy-based heterogeneity preference measure (Definition 3) depends only on the strictly positive components of $p(\mathcal{P})$, and not on their position. This implies that the enumeration in $\mathbf{W}(\mathbf{X})$ is irrelevant to the definition of the entropy-based heterogeneity preference measure. With this in mind, and to shorten notation, the discrete probability distribution of a profile $\mathcal{P}$ will be delimited by parentheses when all components are reported, i.e., when an enumeration of $\mathbf{W}(\mathbf{X})$ is required, $p(\mathcal{P}) = (p_i)_{i=1}^{\phi(k)}$. When only the set of strictly positive components is written and the order is irrelevant, the discrete probability distribution of a profile $\mathcal{P}$ will be delimited by braces, $p(\mathcal{P}) = \{p_{\alpha_i}\}_{i=1}^m$, and the m indices of the strictly positive components of the profile discrete probability distribution will be listed as $\{\alpha_1, \dots, \alpha_m\} \in \{1, 2, \dots, \phi(k)\}$.

Figure 1 shows the sequential process of the proposed approach.

To make the definition concrete, consider the preference profile $\mathcal{P}$ from Example 1 and its associated discrete distribution $p(\mathcal{P})$. Depending on whether an enumeration of $\mathbf{W}(\mathbf{X})$ is fixed, this distribution can be represented in two equivalent ways:

- If an enumeration of $\mathbf{W}(\mathbf{X})$ is required, $p(\mathcal{P})$ is written as a vector in parentheses:

$$p(\mathcal{P}) = \left(0, 0, 0, \frac{2}{5}, 0, 0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0\right).$$

- If no enumeration of $\mathbf{W}(\mathbf{X})$ is required, $p(\mathcal{P})$ is written as a multiset of its nonzero components in braces:

$$p(\mathcal{P}) = \left\{\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right\}.$$

The two representations of $p(\mathcal{P})$ differ in the information they retain. The vector form $(p_i)_{i=1}^{\phi(k)}$ specifies both the probability values and their positions in $\mathbf{W}(\mathbf{X})$, that is, the identities of the complete preorders that are actually present in the profile. By contrast, the braced form retains only the strictly positive probability values, viewed as an unordered multiset, and therefore discards positional information.

For the computation of the entropy-based heterogeneity preference measure δ_H , however, this positional information is irrelevant. Since δ_H depends solely on the multiset of strictly positive probability values, and not on the specific complete preorders in $\mathbf{W}(\mathbf{X})$ to which those values are assigned, both representations lead to the same value of δ_H . Consequently,

$$\delta_H\left((p_i)_{i=1}^{\phi(k)}\right) = \delta_H(\{p_{\alpha_i}\}_{i=1}^m).$$

This equivalence is specific to δ_H and does not extend to heterogeneity measures that are sensitive to the identity of the preorders present in the profile.

Example 1 (continued) Once the entropy-based heterogeneity preference measure has been defined, Equation (2) can be applied to the profile $\mathcal{P}$ of Example 1, whose distribution over the preorder types with strictly positive frequency is

$$p(\mathcal{P}) = \left\{ \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right\}.$$

This yields

$$\delta_H(\mathcal{P}) = -\frac{2}{5} \log \frac{2}{5} - 3 \frac{1}{5} \log \frac{1}{5} = \frac{2}{5} \log \frac{5}{2} + \frac{3}{5} \log 5 \approx 1.3322.$$

The corresponding relative entropy-based heterogeneity preference measure is

$$\delta_{H_R}(\mathcal{P}) = \frac{\delta_H(\mathcal{P})}{\log \phi(3)} = \frac{\delta_H(\mathcal{P})}{\log 13} \approx 0.5194.$$

Having established that the definition is independent of the chosen representation of $p(\mathcal{P})$, it is now shown that the entropy-based index defines a heterogeneity preference measure and satisfies the additional requirements of preference neutrality and preference anonymity.

Lemma 1 *The entropy-based heterogeneity preference measure δ_H is a heterogeneity preference measure. Moreover, it satisfies preference neutrality and preference anonymity.*

Proof First, $\delta_H(\mathcal{P}) \in [0, \infty)$, since it is a sum of non-negative terms. Indeed, for every i such that $p_i > 0$, $0 < p_i \leq 1$ implies $-\log p_i \geq 0$, and hence $-p_i \log p_i \geq 0$.

It remains to prove that the minimum preference heterogeneity condition is satisfied. By definition,

$$\delta_H(\mathcal{P}) = 0 \iff -\sum_{i=1}^{\phi(k)} p_i \log p_i = 0. \quad (3)$$

Since every term $-p_i \log p_i$ is non-negative, the previous equality holds if and only if

$$p_i \log p_i = 0, \quad \forall i = 1, \dots, \phi(k). \quad (4)$$

Thus, for every $i = 1, \dots, \phi(k)$, either $p_i = 0$ or $p_i = 1$. Since $\sum_{i=1}^{\phi(k)} p_i = 1$, there exists an index $j \in \{1, \dots, \phi(k)\}$ such that $p_j = 1$ and $p_i = 0$ for every $i \neq j$. Therefore, all agents in $\mathcal{P}$ express the same complete preorder, and the preference profile is unanimous. The converse implication is immediate, since a unanimous profile induces a degenerate distribution with one component equal to one and all remaining components equal to zero.

Finally, preference neutrality and preference anonymity follow from the fact that permutations of alternatives and permutations of agents only permute the components of the distribution associated with the profile, without changing their values. Since δ_H depends only on the multiset of strictly positive probability values, it follows that

$$\delta_H(\pi\mathcal{P}) = \delta_H(\mathcal{P})$$

for every permutation π of the alternatives, and

$$\delta_H(\mathcal{P}^\sigma) = \delta_H(\mathcal{P})$$

for every permutation σ of the agents. $\square$

Once the proposal has been introduced, the following subsection discusses its properties and behaviour.

4.1 The entropy-based heterogeneity preference measure: properties and behaviour

The main theoretical properties of the entropy-based heterogeneity preference measure are formally defined in this section and summarised in Appendix A, Table A2, while the corresponding proofs are provided in Appendix B. These properties reinforce the theoretical soundness of the measure and clarify its normative and operational interpretation in group decision-making settings. Continuity ensures stability under small changes in the empirical distribution of preferences. Non-negativity and the zero value under unanimity provide a direct interpretation of heterogeneity as departure from complete agreement. The maximum value $\log \phi(k)$ provides a natural benchmark for comparison and normalisation. Alternative invariance ensures that the measure is not affected by unanimously ranked enlargements of the alternative set, whereas replication invariance guarantees invariance under changes in population size that preserve relative frequencies. Convergence to unanimity captures the disappearance of heterogeneity when an increasingly dominant group shares the same preference, whereas grouping formalises the entropy loss associated with merging preference categories.

- (i) **Continuity.** Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile and $(p_i)_{i=1}^{\phi(k)}$ its discrete probability distribution. The entropy-based heterogeneity preference measure, $\delta_H(\mathcal{P})$, can be extended by continuity from $[0, 1]^{\phi(k)}$ to $\mathbb{R}^+$.
- (ii) **Non-negativity.** Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile and $(p_i)_{i=1}^{\phi(k)}$ its discrete probability distribution. The entropy-based heterogeneity preference measure, $\delta_H(\mathcal{P})$ is a strictly positive function when $\mathcal{P}$ is non-unanimous.
- (iii) **Boundedness.** Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile and $(p_i)_{i=1}^{\phi(k)}$ its discrete probability distribution. The maximum value of the entropy-based heterogeneity preference measure is $\log \phi(k)$.

The following property provides the formal expression, within the present preference-heterogeneity framework, of the aggregation (coarse-graining) rationale underlying Shannon entropy. It captures the entropy loss associated with merging two preference categories into a single one. From a practitioner's perspective, this axiom can be interpreted as the action of a moderator who guides the population toward a common agreement by consolidating distinct alternatives. Consequently, Eq. (5) serves to quantify precisely the gain in preference homogeneity, or consensus, resulting from such a deliberation process.

- (iv) **Grouping.** Suppose a preference profile $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ and two disjoint sets $\{I_1, I_2\}$ of $\mathbf{N}$ with cardinalities $|I_1| = n_1$ and $|I_2| = n_2$, respectively. These two sets induce the partition $\{I_1, I_2, \mathbf{N} \setminus (I_1 \cup I_2)\}$. It is assumed that the agents' opinions on $\mathbf{X}$ are unanimous within each set, but different between them. Let $\mathcal{P}^{I_1} = (\mathcal{R}_1, \dots, \mathcal{R}_1) \in \mathbf{P}(\mathbf{X})$ be the preference subprofile of the agents in I_1 , where $\mathcal{R}_1$ is their unanimous preference on $\mathbf{X}$, and $\mathcal{P}^{I_2} = (\mathcal{R}_2, \dots, \mathcal{R}_2) \in \mathbf{P}(\mathbf{X})$ be the preference subprofile of the agents in I_2 , where $\mathcal{R}_2$ is their unanimous preference on $\mathbf{X}$, such that $\mathcal{P}^{I_1} = (\mathcal{R}_1, \dots, \mathcal{R}_1) \neq \mathcal{P}^{I_2} = (\mathcal{R}_2, \dots, \mathcal{R}_2)$. Moreover, agents in $\mathbf{N} \setminus (I_1 \cup I_2)$ have opinions different from $\mathcal{R}_1$ and $\mathcal{R}_2$.

In this case, if all agents included in I_2 change their preferences on $\mathbf{X}$ so that they unanimously abandon $\mathcal{R}_2$ and unanimously adopt $\mathcal{R}_1$, generating the preference profile $\overline{\mathcal{P}}$, the entropy-based heterogeneity preference measure of $\overline{\mathcal{P}}$ is given by:

$$\delta_H(\overline{\mathcal{P}}) = \delta_H(\mathcal{P}) - \frac{n_1 + n_2}{n} \delta_H(\mathcal{P}^{I_1 \cup I_2}). \quad (5)$$

- (v) **Alternative invariance.** Let $\mathbf{X} = \{x_1, \dots, x_k\}$ be a set of alternatives and $\widetilde{\mathbf{X}}^q = \{x_1, \dots, x_k, x_{k+1}, \dots, x_{k+q}\}$ be its expanded set. Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile, and let $\widetilde{\mathcal{P}}^q = (\widetilde{\mathcal{R}}_1, \dots, \widetilde{\mathcal{R}}_n) \in \mathbf{P}(\widetilde{\mathbf{X}})$ be any expanded preference profile such that the new alternatives are unanimously ordered by all agents, that is, $\widetilde{\mathcal{R}}_i|_{\{x_{k+1}, \dots, x_{k+q}\}} = \widetilde{\mathcal{R}}_j|_{\{x_{k+1}, \dots, x_{k+q}\}}$, for all $i, j \in \{1, \dots, n\}$. Then, the entropy-based heterogeneity preference measure of the expanded preference profile $\widetilde{\mathcal{P}}^q$ is equal to the original one. Formally,

$$\delta_H(\mathcal{P}) = \delta_H(\widetilde{\mathcal{P}}^q). \quad (6)$$

- (vi) **Replication invariance.** Let $\mathcal{P} \in \mathbf{W}(\mathbf{X})^n$ be a preference profile and $\mathcal{P} \uplus \mathcal{P} \in \mathbf{W}(\mathbf{X})^{2n}$ be a replication of $\mathcal{P}$. The entropy-based heterogeneity preference measure does not change when the preference profile $\mathcal{P}$ is replicated. Formally,

$$\delta_H(\mathcal{P}) = \delta_H(\mathcal{P} \uplus \mathcal{P}). \quad (7)$$

- (vii) **Convergence to unanimity.** This property establishes that if q new agents are repeatedly introduced to $\mathbf{N}$ such that their preferences on $\mathbf{X}$ coincide, the entropy-based heterogeneity preference measure of the extended preference profile approaches zero as q tends to infinity. Formally,

Let $\mathbf{N}$ be an agent set and $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{W}(\mathbf{X})^n$ be a preference profile associated with $\mathbf{N}$. Suppose that q new agents are added to $\mathbf{N}$, $\bar{\mathbf{N}}^q = \{1, \dots, n, n+1, \dots, n+q\}$, and suppose that the extension of $\mathcal{P}$ associated with $\bar{\mathbf{N}}^q$, $\bar{\mathcal{P}}^q = (\mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{R}_{n+1}, \dots, \mathcal{R}_{n+q}) \in \mathbf{W}(\mathbf{X})^{n+q}$, is defined such that $\mathcal{R}_n = \mathcal{R}_{n+1} = \dots = \mathcal{R}_{n+q}$. Then,

$$\lim_{q \rightarrow \infty} \delta_H(\bar{\mathcal{P}}^q) = 0. \quad (8)$$

Remark 6 The grouping property allows the decrease, or loss, of entropy to be quantified when two groups of individuals reach a unanimous decision. This property will be key to characterising Shannon entropy in the following section. Although other forms of entropy have been proposed in the literature, e.g., Tsallis and Rényi [69], their structures do not readily allow for the measurement of this effect. Consequently, they are excluded from the present study.

4.2 Necessary and sufficient conditions for the entropy-based heterogeneity preference measure

In the vein of established axiomatic traditions in Social Choice Theory, an axiomatic characterisation of the entropy-based heterogeneity preference measure is provided in this section, with the aim of establishing it as a theoretically sound and uniquely determined index. The proof strategy adapts the classical characterisation of the information function due to Tverberg [70], which establishes that any real-valued function on finite probability distributions satisfying continuity, symmetry, and a specific recursive decomposition property must be proportional to Shannon's entropy.

It should be noted, however, that the contribution of the present result does not reside in the proof technique itself, which is adapted from Tverberg's original derivation, but in its systematic transposition to the domain of preference heterogeneity measurement over complete preorders. This transposition involves three elements that are entirely absent from Tverberg's result: (i) the construction of a probability space over the set of all complete preorder types induced by an observed preference profile, which

requires the structural apparatus developed in Sections 3 and 4; (ii) the formulation and normative justification of domain-specific axioms, particularly alternative invariance and the grouping property (iv), whose interpretation is grounded in the structural properties of collective preference profiles rather than in abstract information-theoretic considerations; and (iii) the proof that this specific combination of axioms uniquely characterises δ_H within the class of continuous distribution-based heterogeneity measures defined over complete preorders, a result that had not previously been established in the preference measurement literature.

Theorem 2 *Let $\delta : \mathbf{P}(\mathbf{X}) \rightarrow [0, \infty)$ be a heterogeneity preference measure on $\mathcal{P}$. Then, $\delta = \delta_H$ if and only if δ verifies (i)-continuity, (iii)-boundedness, and (iv)-grouping.*

A detailed proof of Theorem 2 can be found in Appendix C. In what follows, the main ideas underlying the argument are presented.

Outline of the proof

The necessity of the result follows directly from the preceding sections. The proof of sufficiency is developed in three stages. In the first stage, the problem is reduced to the case where agents express binary opinions. Under this assumption, the discrete probability distribution depends on a single parameter, namely the probability of success p . The aim is therefore to identify a function, interpreted as a measure of preference heterogeneity, that depends on a single variable. The combination of properties (i) and (iv) leads to a differential equation, which admits an analytical solution up to a multiplicative constant. Although straightforward, the derivation involves technical computations and requires some elements of analysis. In the second stage, a simple inductive argument, in which property (iv) plays a crucial role, extends the measure to the general case. In the final stage, the value of the constant is determined using property (iii).

5 Empirical and simulation analysis

This section evaluates the proposed entropy-based heterogeneity measure through three complementary exercises. First, the measure is applied to ten ranked-choice elections obtained from *PrefLib: A Library for Preferences*⁵. These empirical cases illustrate the behaviour of the index in real preference profiles with different electorate sizes, numbers of alternatives, observed preorder supports, and preference distributions.

Second, a complementary comparison with selected consensus- and distance-based indicators is provided for the same election profiles. Since these measures are originally formulated as consensus indices, they are transformed into dissent indicators to make their orientation comparable with the entropy-based heterogeneity measure.

⁵<https://www.preflib.org>

The comparison is intended to provide empirical intuition about similarities and differences between the proposed information-theoretic index and existing reference- or distance-based approaches, rather than to constitute an exhaustive benchmarking exercise.

Third, simulation experiments are conducted under controlled distributional patterns. These experiments complement the real election analysis by assessing how the measure reacts to changes in concentration, multimodality, skewness, and the number of distinct preorder types.

5.1 Case Studies: Real Elections

The empirical analysis is based on ten elections conducted under *ranked-choice voting* rules, selected from the PrefLib electoral preference database (<https://www.preflib.org>). The primary motivation behind the selection of these specific ten datasets was to ensure data completeness while covering a wide structural variety, characterised by different numbers of alternatives (k) and substantially varying electorate sizes (n). This structural diversity makes it possible to demonstrate that the underlying entropy-based heterogeneity calculation is robust and not inherently driven by sample size.

When using the PrefLib database for the cases considered, structures are observed where voters can select multiple candidates per position, and candidates can appear repeatedly across different positions. This generates voting profiles that include rankings and ties (represented within braces), as well as repetitions that suggest a higher intensity of approval (e.g., A, B, C, D ; $A, \{B, C\}, D$; $\{A, B\}, \{A, B\}, \{A, B\}, \{C, D\}$ or $B, B, B, \{A, C, D\}$). In this work, these profiles are transformed into preorders by ignoring this intensity. Specifically, the aforementioned cases are converted into $A \succ B \succ C \succ D$, $A \succ B \sim C \succ D$, $A \sim B \succ C \sim D$, and $B \succ A \sim C \sim D$, respectively. This methodology implies a deliberate simplification of the original information, ignoring candidate redundancy to focus exclusively on the hierarchy of the preference order. Consequently, after this preprocessing step, all structures are successfully mapped, and each retained ballot is interpreted as a complete preorder over the set of alternatives, that is, a ranking that may involve indifference (ties) between candidates without introducing informational bias. Indifference is indeed present in the data: in several elections the number $m(\mathcal{P})$ of observed preorder types exceeds the $k!$ strict linear orders, which confirms that the empirical profiles genuinely exploit the complete-preorder structure rather than reducing to linear orders. The aim is to assess whether the proposed measure remains informative when the number of voters is large and the theoretical space of complete preorders grows rapidly with the number of candidates.

Table 4 reports the elections included in the analysis, grouped by the number of candidates. Two cases are considered for each value of $k \in \{4, 5, 7, 9, 11\}$, allowing comparisons between elections with the same number of alternatives. The table also reports the cardinality of $\mathbf{W}(\mathbf{X})$, which increases sharply with k and illustrates the rapid growth of the theoretical preference space.

The main empirical results are reported in Table 5. The table includes the number of voters, the number $m(\mathcal{P})$ of distinct complete preorders observed in each profile,

Table 4: Real examples included in this contribution.

Number of candidates	Cardinality of $\mathbf{W}(\mathbf{X})$	Real election examples
4	75	2010 Berkeley City Council 2009 Pierce County Auditor
5	541	2000 American Psychological Association 2008 Pierce County Executive
7	47,293	2009 Minneapolis Board of Estimate and Taxation 2008 Pierce County Assessor Treasurer
9	7,087,261	2009 Minneapolis Park and Recreation Commissioner 2012 San Francisco Board of Supervisors
11	1,622,632,573	2010 Oakland Mayoral 2007 Glasgow City Council

the entropy-based heterogeneity measure $\delta_H(\mathcal{P})$, and its normalised version $\delta_{H_R}(\mathcal{P})$. These metrics allow for meaningful comparisons across different types of electorates. For instance, the 2000 American Psychological Association (APA) election is particularly relevant in this context: this dataset involves a highly specialised professional organisation where a greater degree of consensus or structured agreement is conceptually expected, contrasting sharply with a strongly polarised municipal general election such as the 2010 Oakland mayoral race.

In addition, to clarify the interpretation of the relative index, the table also reports a decomposition of $\delta_{H_R}(\mathcal{P})$ into an observed-support component and a coverage component.

The interpretation of $\delta_{H_R}(\mathcal{P})$ requires particular care. This index is normalised by $\log \phi(k)$, where $\phi(k)$ denotes the total number of complete preorders on k alternatives. Since $\phi(k)$ grows very rapidly with k , a lower value of $\delta_{H_R}(\mathcal{P})$ for larger k may partly reflect the increasing size of the theoretical preference space, rather than a lower degree of heterogeneity among the observed preorder types alone.

To disentangle these two effects, the following decomposition is considered:

$$\frac{\delta_H(\mathcal{P})}{\log \phi(k)} = \frac{\delta_H(\mathcal{P})}{\log m(\mathcal{P})} \cdot \frac{\log m(\mathcal{P})}{\log \phi(k)},$$

where $m(\mathcal{P})$ denotes the number of distinct complete preorders observed in the profile. For $m(\mathcal{P}) > 1$, the first factor, $\delta_H(\mathcal{P})/\log m(\mathcal{P})$, provides a support-conditional ratio that measures the evenness of the observed preference distribution relative to the preorder types actually represented in the data. The second factor quantifies the logarithmic coverage of the complete theoretical preorder space by the observed support.

The distinction between these two components is important. A value of $\delta_H(\mathcal{P})/\log m(\mathcal{P})$ close to one indicates that the probability mass is distributed relatively evenly across the preorder types actually observed. However, this ratio does not indicate how large the observed support is relative to the full space $\mathbf{W}(\mathbf{X})$. That complementary information is provided by $\log m(\mathcal{P})/\log \phi(k)$.

In the empirical cases considered, the support-conditional ratio $\delta_H(\mathcal{P})/\log m(\mathcal{P})$ remains relatively high across all ten elections, with values between approximately 0.63 and 0.89. By contrast, the coverage term $\log m(\mathcal{P})/\log \phi(k)$ decreases markedly as k increases. Consequently, the lower values of $\delta_{H_R}(\mathcal{P})$ observed for larger candidate sets are mainly associated with the rapid growth of the complete preorder space rather than with an extreme concentration of the observed preferences.

Accordingly, $\delta_{H_R}(\mathcal{P})$ should be interpreted as a full-space relative heterogeneity index, whereas $\delta_H(\mathcal{P})/\log m(\mathcal{P})$ provides a complementary assessment of the evenness of the preference distribution conditional on the support actually observed.

Table 5: Summary of real examples and decomposition of the relative entropy index.

k	Real examples	Voters	$m(\mathcal{P})$	$\delta_H(\mathcal{P})$	$\delta_{H_R}(\mathcal{P})$	$\frac{\delta_H(\mathcal{P})}{\log m(\mathcal{P})}$	$\frac{\log m(\mathcal{P})}{\log \phi(k)}$
4	2010 Berkeley City Council election	4,189	39	2.63	0.61	0.718	0.849
	2009 Pierce County Auditor election	153,721	45	2.41	0.56	0.633	0.882
5	2000 American Psychological Association election	20,239	205	4.72	0.75	0.887	0.846
	2008 Pierce County Executive election	298,788	99	3.35	0.53	0.729	0.730
7	2009 Minneapolis Board of Estimate and Taxation election	32,086	210	3.69	0.34	0.690	0.497
	2008 Pierce County Assessor Treasurer election	262,312	296	4.46	0.41	0.784	0.529
9	2009 Minneapolis Park and Recreation Commissioner election	36,655	470	4.79	0.30	0.779	0.390
	2012 San Francisco Board of Supervisors election	35,183	436	5.15	0.33	0.847	0.385
11	2010 Oakland Mayoral election	119,962	1,090	4.90	0.23	0.701	0.330
	2007 Glasgow City Council election	8,803	2,202	5.74	0.27	0.746	0.363

Figure 2 provides a visual representation of the two complementary perspectives on relative preference heterogeneity introduced above. In both panels, each point corresponds to an election, the horizontal axis reports the number of distinct complete

preorders observed, $m(\mathcal{P})$, and bubble size is proportional to the number of voters. Panel (a) displays the relative entropy-based heterogeneity measure $\delta_{H_R}(\mathcal{P})$, whereas Panel (b) displays the ratio $\delta_H(\mathcal{P})/\log m(\mathcal{P})$.

The comparison between the two panels clarifies the effect of the rapidly expanding theoretical preference space. While $\delta_{H_R}(\mathcal{P})$ tends to attain lower values in elections with larger candidate sets, the values of $\delta_H(\mathcal{P})/\log m(\mathcal{P})$ remain relatively high across all ten elections. Consequently, the lower values of $\delta_{H_R}(\mathcal{P})$ observed for the higher-cardinality cases should not be interpreted as evidence of a strongly concentrated distribution over the preorder types actually observed. Instead, they are largely associated with the decreasing coverage of the complete-preorder space, as captured by the term $\log m(\mathcal{P})/\log \phi(k)$. The two panels therefore provide complementary information: Panel (a) evaluates heterogeneity relative to the full set of theoretically admissible complete preorders, whereas Panel (b) evaluates the evenness of the preference distribution conditional on the support observed in the data.

While the values in Table 5 allow for comparative assessments, they do not provide absolute thresholds for classifying heterogeneity levels across cases. Establishing such benchmarks would require a broader empirical calibration and constitutes a relevant direction for future research. Nevertheless, the results illustrate the scalability and interpretability of the proposed measure in large preference profiles. In particular, the measure remains computable and informative despite two central challenges in preference aggregation: the rapid growth of the set of complete preorders as k increases and the large cardinality of the agent set.

5.2 Comparison with consensus- and distance-based dissent indicators

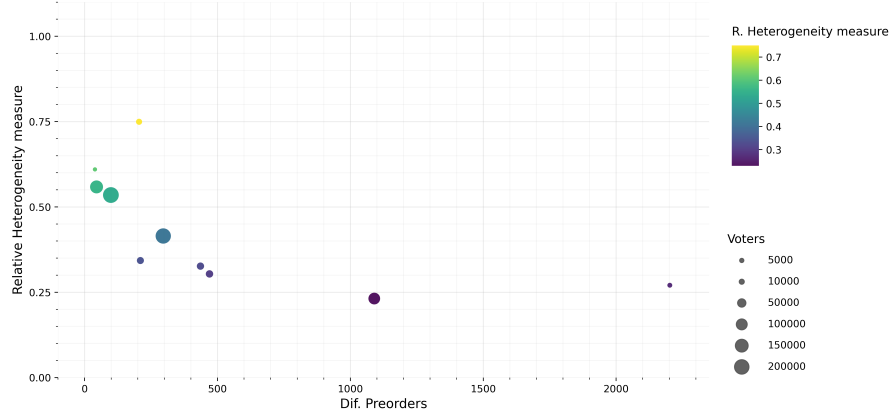
To further clarify the empirical behaviour of the entropy-based heterogeneity measure, a focused comparison with established consensus-based approaches was conducted. Two families of indicators were considered. First, the *reference consensus measure* of [23] was computed using the Borda and Copeland reference rankings. Since this measure is formulated as a consensus indicator, attaining larger values as the profile approaches the corresponding reference ranking and taking values in the unit interval, the associated *reference dissent measure* was defined, in order to align its orientation with the entropy-based index, as

$$RDM = 1 - RCM \in [0, 1].$$

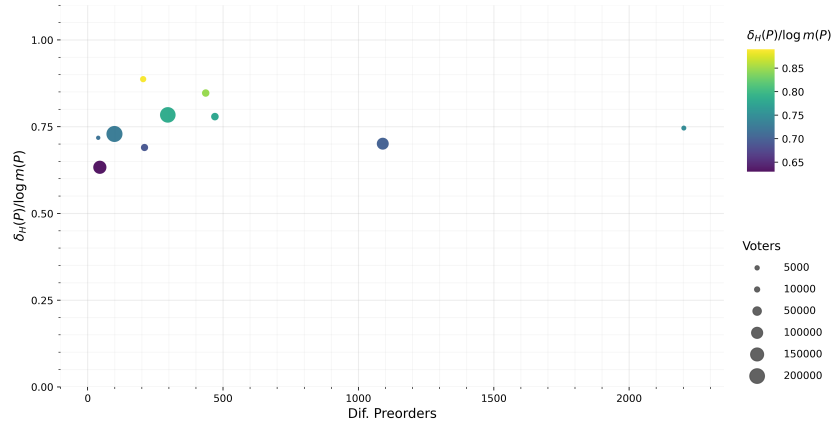
Two variants were obtained, the Borda- and Copeland-based measures, denoted by RDM_B and RDM_C , respectively.

Second, the consensus measure proposed in [19] was also considered. In that contribution, consensus is evaluated through the distance between complete preorders, and a family of such measures is examined within a common framework according to the distance employed. To express the results in terms of dissent rather than consensus, the transformation

$$D = 1 - M \in [0, 1]$$



(a) Relative entropy-based heterogeneity measure $\delta_{H_R}(\mathcal{P})$, normalised with respect to the full space of complete preorders.



(b) Ratio $\delta_H(\mathcal{P})/\log m(\mathcal{P})$, evaluated relative to the complete preorder types actually observed in the profile.

Figure 2: Summary of the real election examples under two complementary relative entropy perspectives. Each bubble denotes one election. In both panels, the x -axis reports the number of distinct complete preorders observed, $m(\mathcal{P})$, bubble size is proportional to the number of voters, and colour intensity represents the quantity displayed on the corresponding y -axis. Panel (a) reports $\delta_{H_R}(\mathcal{P})$, whereas Panel (b) reports $\delta_H(\mathcal{P})/\log m(\mathcal{P})$.

was applied to each measure, where M denotes the corresponding consensus measure, normalised to the unit interval. The six resulting dissent indicators correspond to the distances considered in [19]: the discrete distance, the Manhattan distance (ℓ_1), the Euclidean distance (ℓ_2), the Chebyshev distance (ℓ_∞), the cosine distance, and the

Kemeny distance, and are denoted by D_{disc} , D_{Manh} , D_{Euc} , D_{Cheb} , D_{Cos} , and D_{Kem} , respectively.

This normalisation makes all indicators directionally comparable: larger values of $\delta_{H_R}(\mathcal{P})$, RDM , and D reflect, respectively, greater heterogeneity, dissent, or disagreement. The comparison should nonetheless be read with caution, since the indices rest on different principles. The entropy-based measure summarises the informational dispersion of the empirical distribution of complete preorder types; RDM evaluates disagreement relative to a single reference ranking; and the D -indicators quantify a distance-based departure from consensus under a specific representation of complete preorders.

Table 6: Comparison between the relative entropy-based heterogeneity measure and alternative dissent indicators.

k	Real election	Entropy	Reference dissent		Distance-based dissent					
		δ_{H_R}	RDM_B	RDM_C	D_{disc}	D_{Manh}	D_{Euc}	D_{Cheb}	D_{Cos}	D_{Kem}
4	Berkeley City Council, 2010	0.6099	0.6815	0.6815	0.9092	0.4125	0.4548	0.5079	0.2611	0.3257
	Pierce County Auditor, 2009	0.5586	0.6842	0.6842	0.8852	0.3713	0.4112	0.4560	0.2218	0.2958
5	American Psychological Association, 2000	0.7494	0.5205	0.5205	0.9835	0.5861	0.5992	0.6769	0.4082	0.4561
	Pierce County Executive, 2008	0.5323	0.4880	0.4880	0.9464	0.4979	0.5207	0.6014	0.3272	0.3871
7	Minneapolis Board of Estimate and Taxation, 2009	0.3429	0.3776	0.3776	0.9223	0.3757	0.4327	0.5434	0.2367	0.2884
	Pierce County Assessor Treasurer, 2008	0.4148	0.3750	0.3759	0.9756	0.4620	0.5175	0.6248	0.3113	0.3639
9	Minneapolis Park and Recreation Commissioner, 2009	0.3037	0.3091	0.3092	0.9765	0.3918	0.4744	0.6011	0.2672	0.3105
	San Francisco Board of Supervisors, 2012	0.3265	0.3633	0.3633	0.9876	0.3936	0.4843	0.6087	0.2739	0.3167
11	Oakland Mayoral election, 2010	0.2315	0.3531	0.3538	0.9820	0.2962	0.4076	0.5596	0.2036	0.2385
	Glasgow City Council, 2007	0.2705	0.3184	0.3184	0.9822	0.3959	0.4645	0.5916	0.2609	0.3093

The comparison reported in Table 6 reveals several patterns. First, the Borda- and Copeland-based Reference Dissent Measures are virtually indistinguishable across

the ten examples. The reference-based dissent diagnosis is therefore highly robust to the choice between Borda and Copeland as the underlying scoring rule, and its interpretation is not driven by a particular reference procedure.

Second, the low-cardinality elections provide a transparent benchmark in which the indicators agree. For the four-candidate cases, where $\phi(4) = 75$ and the indicators can be inspected directly, the 2010 Berkeley City Council and 2009 Pierce County Auditor elections exhibit comparatively high $\delta_{H_R}(\mathcal{P})$ (0.6099 and 0.5586) together with the highest reference dissent of the whole collection ($RDM_B \approx 0.68$). In these tractable settings the entropy-based index, the reference dissent, and the Kemeny-based dissent deliver a coherent qualitative diagnosis, offering an easily verifiable point of contact with the existing literature.

Third, as the analysis moves to richer configurations, the indicators dissociate in an informative way. The 2000 American Psychological Association election attains the highest relative entropy of the collection ($\delta_{H_R}(\mathcal{P}) = 0.7494$) together with the highest distance-based dissent values ($D_{\text{Cheb}} = 0.6769$, $D_{\text{Kem}} = 0.4561$), yet its reference dissent is only intermediate ($RDM_B = 0.5205$), below that of the four-candidate elections. This follows from the underlying principles: a profile may remain only moderately distant from a single reference ranking, keeping RDM bounded, while distributing its mass over many distinct preorder types, which raises $\delta_{H_R}(\mathcal{P})$. The entropy-based index is therefore broadly coherent with, but not equivalent to, the reference-ranking view of disagreement. For the higher-cardinality elections ($k = 9, 11$), the comparatively low values of $\delta_{H_R}(\mathcal{P})$ should be read in the light of the decomposition introduced in Section 5.1 (Table 5), where they are shown to stem mainly from the coverage term $\log m(\mathcal{P}) / \log \phi(k)$ rather than from an extreme concentration of the observed preferences.

Fourth, the discrete dissent indicator D_{disc} takes very high values in all examples. Since exact coincidence between complete preorders is required, small ordinal differences and large structural differences are treated alike, so D_{disc} acts as a rough diagnostic of non-coincidence but offers limited discriminatory power among the real profiles considered here. The remaining distance-based indicators provide a more graded diagnosis and display greater inter-case variation. Among them, the Kemeny-based indicator is the most meaningful from an ordinal standpoint, whereas the Manhattan, Euclidean, Chebyshev, and cosine versions depend more directly on the numerical representation of complete preorders adopted in [19].

Figure 3 provides a guided visual summary of the same comparison. The figure should not be read as a ranking device, but as a profile comparison tool. The entropy-based axis, $\delta_{H_R}(\mathcal{P})$, offers the natural starting point. The two reference dissent axes, RDM_B and RDM_C , may then be inspected to verify their close correspondence. Next, the uniformly high values of D_{disc} highlight the limited discriminatory power of the discrete metric. Finally, the Manhattan, Euclidean, Chebyshev, cosine, and Kemeny axes reveal the finer differences between elections under alternative metric representations of dissent.

Overall, the comparison supports a complementary reading of the proposed measure. Consensus- and distance-based indicators evaluate disagreement relative to a

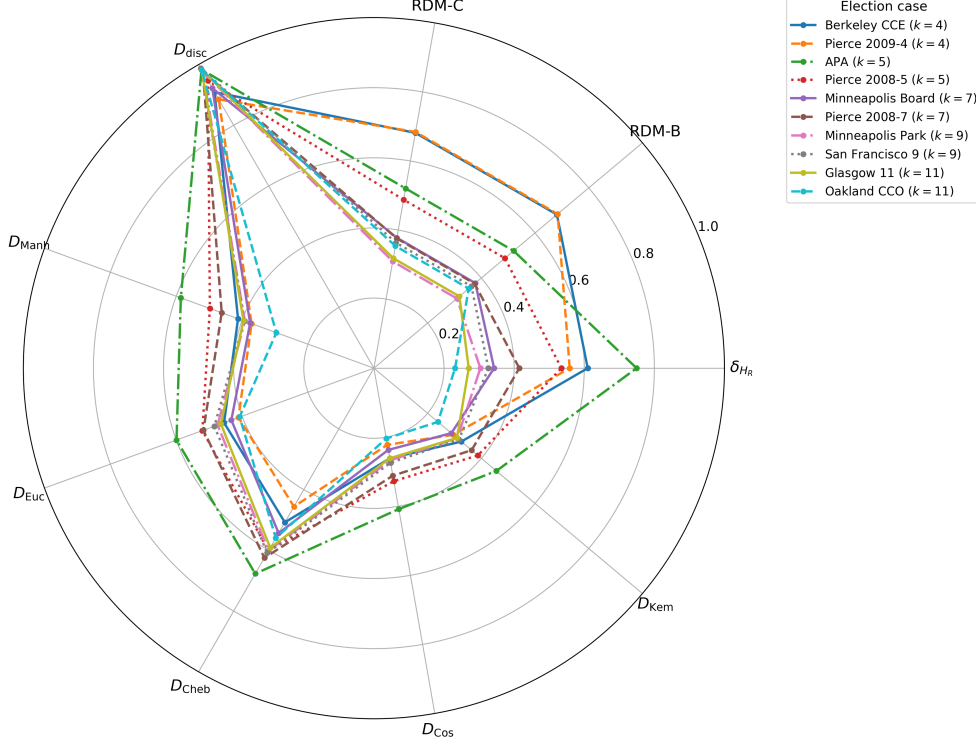


Figure 3: Radar chart comparing the relative entropy-based heterogeneity measure and alternative dissent indicators for the ten real election examples. Each polygon corresponds to one election. All indicators are normalised in the unit interval. The figure highlights the near coincidence between the Borda- and Copeland-based reference dissent measures, the systematically high values of the discrete dissent indicator, and the more graded behaviour of the remaining distance-based measures.

specific reference ranking or metric structure, whereas the entropy-based measure captures the uncertainty associated with the empirical frequency distribution of complete preorder types. The two families also differ in their computational requirements: the reference- and distance-based indicators rely on a numerical representation of preorders and on the evaluation of distances between them, whereas $\delta_H(\mathcal{P})$ reduces to counting type frequencies. The proposed index therefore conveys information that is not fully captured by existing consensus-based indicators while remaining computationally suited to large-scale group decision-making settings.

5.3 Simulation experiments

Simulations have played a significant role in Social Choice Theory for quite some time. Arrow’s pioneering efforts resulted in the first simulated outcomes, illustrating the

impossibility of satisfying all fairness criteria in a voting system, as demonstrated by Arrow’s Impossibility Theorem [57, 71]. Beyond these early studies, many other researchers have conducted simulations to investigate different elements of the decision process. For example, [72] explored the computational aspects of voting systems, [73] conducted a detailed analysis of preference aggregation, while [74] examined the effectiveness of various voting systems through the use of simulated data. In a similar vein, [75] and [76] employed simulations to examine the resilience and vulnerability to strategic manipulation of several voting rules. [77] made another important contribution by introducing the *Majority Judgment* method and comparing its effectiveness with traditional voting systems through simulations, and [78] used simulations to assess how different preference models influence decision outcomes. More recently, [79] analysed the implications of different decision rules using extensive simulations.

These studies underscore the relevance of simulation as a methodological tool for analysing collective decision-making under controlled assumptions. In the present subsection, simulations are used to examine the behaviour of the entropy-based heterogeneity preference measure when the distribution of preferences over complete preorder types changes.

The objective is not to reproduce a specific electoral context, but to assess how the proposed measure reacts to alternative generative patterns. In particular, the analysis focuses on how δ_H varies with the number of distinct preorder types and with the degree of concentration or dispersion of the induced frequency distribution.

The proposed experimental strategy consists of four main steps, which are described below. Figure 5 summarises the experimental strategy.

1. **Selection of the probability distribution.** Eight probability distributions are considered to generate different patterns of concentration and dispersion over preorder indices. Their density functions and parameters are reported in Figure 4 and Table 7. The “2 Modes” and “3 Modes” cases are mixtures of normal distributions and are used to represent preference profiles with two or three regions of concentration, respectively.
2. **Generation of random samples.** For each distribution, a sample $\mathcal{M}$ of N pseudo-random numbers is generated on the unit interval. In all simulations, the sample size is fixed at $N = 10,000$ in order to obtain a stable empirical representation of the underlying distribution.
3. **Generation of preorder samples.** The numerical sample $\mathcal{M}$ is transformed into a sample of complete preorders over $\mathbf{X} = \{x_1, \dots, x_k\}$. For a fixed value of j , with $j \leq \phi(k)$, a support of j preorder types is considered. The unit interval is partitioned into j ordered subintervals, and each pseudo-random value is assigned to the preorder type associated with the subinterval in which it falls. Thus, the continuous generating distribution induces a frequency vector over the j preorder types. The resulting random profile is denoted by $\mathcal{P}_{N,j}$.

The purpose of this step is not to analyse the identity of the preorders themselves, but the frequency distribution induced over the preorder types. This

is sufficient for computing δ_H , since the measure depends on the empirical frequencies and is invariant under relabelling of preorder types.

4. **Computation of the entropy-based heterogeneity measure.** Finally, the empirical frequency vector associated with $\mathcal{P}_{N,j}$ is used to compute the entropy-based heterogeneity preference measure $\delta_H(\mathcal{P}_{N,j})$.

Table 7: Summary of probability distributions used in the simulation. Additional details are provided in Figure 4.

Distribution	Parameters	Simulation	Interpretation
Uniform	a, b $b \geq a$	$U(0, 1)$	All preorder indices are generated with equal probability. This case represents the maximum-dispersion benchmark.
Normal	μ σ	$N(0.5, 0.1)$ $N(0.5, 0.15)$	Preferences concentrate around a central region of the preorder index. The larger standard deviation produces a broader effective spread.
MultiModal Mixture of two normal distributions	(ω_1, ω_2) (μ_1, μ_2) (σ_1, σ_2)	Two peaks: $(0.5, 0.5)$ $(0.25, 0.75)$ $(0.05, 0.05)$	Preferences concentrate around two regions. In the selected parametrisation, the peaks are relatively wider than in the three-mode case.
MultiModal Mixture of three normal distributions	$(\omega_1, \omega_2, \omega_3)$ (μ_1, μ_2, μ_3) $(\sigma_1, \sigma_2, \sigma_3)$	Three peaks: $(1/3, 1/3, 1/3)$ $(0.2, 0.5, 0.8)$ $(0.01, 0.01, 0.01)$	Preferences concentrate around three narrow regions. Although this distribution has more modes, its smaller standard deviations generate a more concentrated effective support.
Beta	$\alpha > 0$ $\beta > 0$	$B(0.1, 0.1)$	Preferences concentrate near the extremes of the preorder index, generating a highly uneven frequency distribution.
Log-normal	$\mu \in (-\infty, +\infty)$ $\sigma > 0$	$\mu = -3, \sigma = 0.7$	A skewed distribution in which a substantial part of the probability mass is concentrated near low preorder indices.
Weibull	$\lambda \in (0, +\infty)$ $k \in (0, +\infty)$	$Weibull(0.1, 2)$	A skewed distribution that generates concentration near a limited region of the preorder index, but with a broader effective spread than the log-normal and beta cases.

For the simulation, the number of alternatives is fixed at $k = 5$, so that $\phi(5) = 541$. For each generating distribution, the parameter j is varied from 2 to $\phi(5)$, and a profile $\mathcal{P}_{N,j}$ is generated with $N = 10,000$ agents. This design makes it possible to observe how the entropy-based heterogeneity measure changes when the number of available preorder types increases and when the induced frequency distribution becomes more or less concentrated.

The histograms and box plot are useful for interpreting the entropy curves, because they show how each continuous generating distribution is translated into an

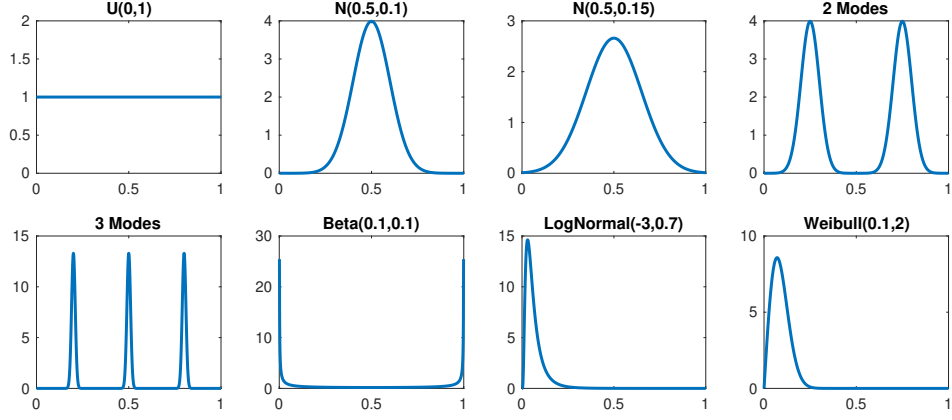


Figure 4: Probability density functions used to generate the preorder samples.

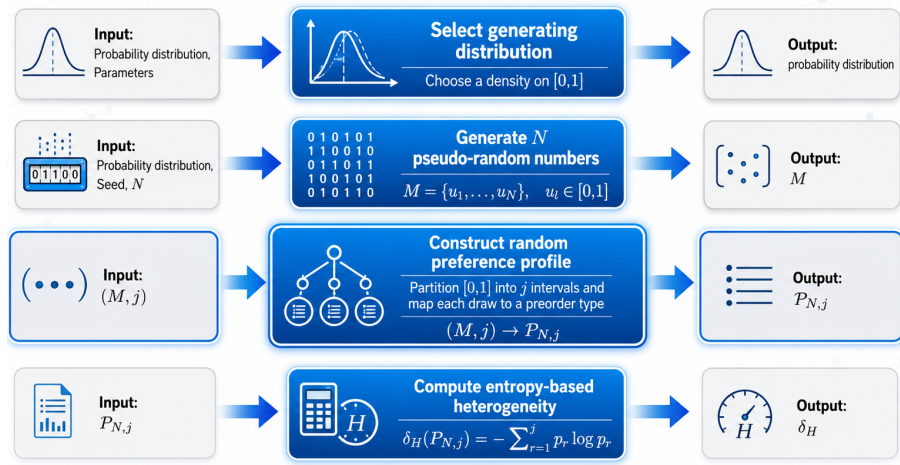


Figure 5: Workflow of the simulation procedure, to be read from left to right. A generating distribution is first selected, then used to produce a numerical sample $\mathcal{M}$, which is mapped into j preorder types to construct the profile $\mathcal{P}_{N,j}$. Finally, the entropy-based heterogeneity measure δ_H is computed.

empirical distribution over preorder indices. Figure 6 reports the histogram of preorder indices for the case $j = \phi(5) = 541$, while Figure 7 summarises the distributional variability of the simulated preorder indices. For instance, the “2 Modes” distribution has fewer peaks than the “3 Modes” distribution, but its peaks are wider because

the standard deviations are $(0.05, 0.05)$ rather than $(0.01, 0.01, 0.01)$. Consequently, in this parametrisation, the “2 Modes” case generates a broader effective spread over preorder types than the “3 Modes” case.

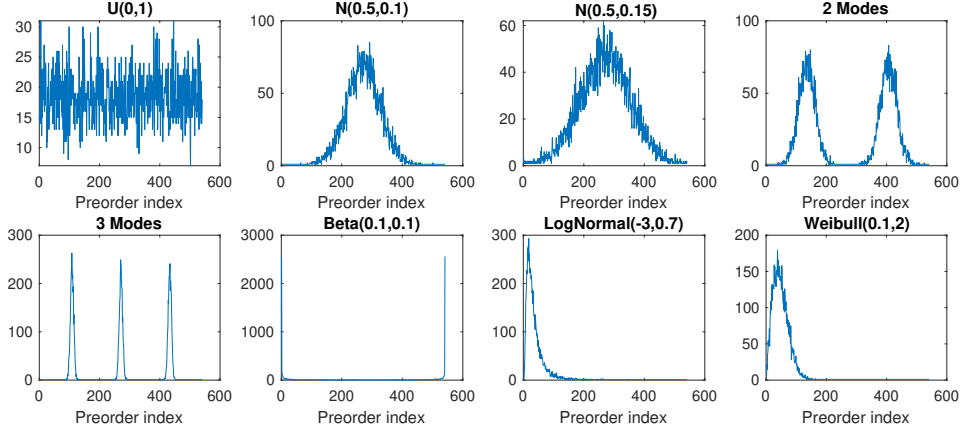


Figure 6: Histogram of preorder indices for $j = \phi(5) = 541$.

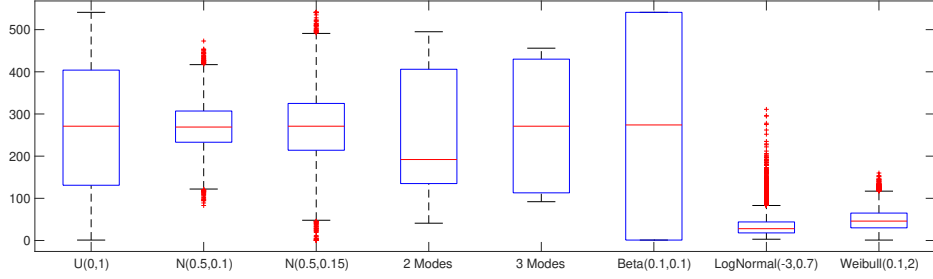


Figure 7: Box-plot analysis of distributional variability across simulated samples for $j = \phi(5) = 541$.

Figure 8 shows the value of $\delta_H(\mathcal{P}_{N,j})$ as a function of j . The maximum benchmark is shown together with the entropy curves induced by the different generating distributions. The uniform distribution $U(0, 1)$ lies on, or very close to, this benchmark, as expected, because it induces approximately balanced frequencies over the available preorder types. More concentrated distributions produce lower entropy values because their empirical frequencies are less evenly spread.

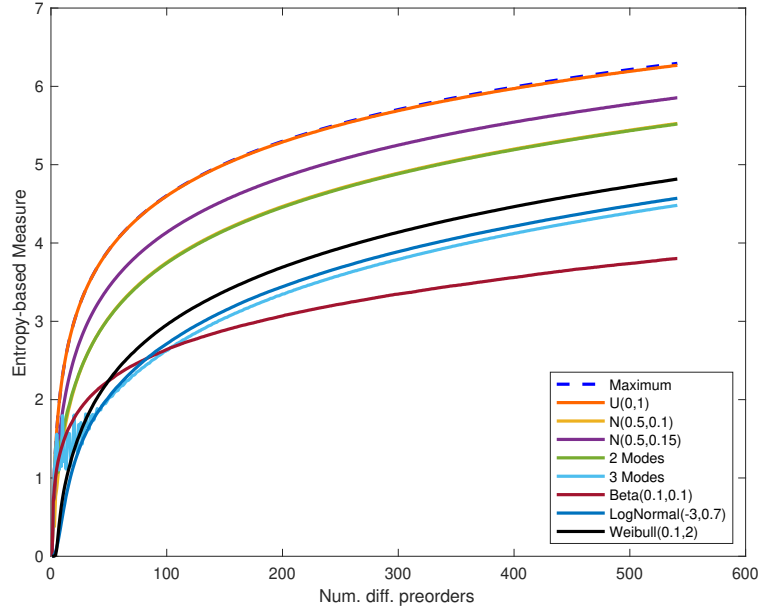


Figure 8: Entropy-based heterogeneity measure for simulated profiles with $k = 5$ alternatives as a function of the number of distinct preorder types. The uniform distribution lies on, or very close to, the maximum benchmark, whereas more concentrated generating distributions produce lower entropy values.

For the selected parametrisation, the following approximate ordering of entropy levels is observed across most of the simulated range:

$$U(0, 1) > N(0.5, 0.15) > 2 \text{ Modes} \approx N(0.5, 0.1) > \text{Weibull} > \text{LogNormal} > 3 \text{ Modes} > \text{Beta}$$

This ordering should not be interpreted as a general hierarchy of distribution families, nor as a strict pointwise ranking for every value of j . Rather, it reflects the effective spread and evenness of the preorder-frequency distributions induced by the specific parameters used in the experiment.

In particular, the fact that the “2 Modes” curve lies above the “3 Modes” curve is explained by the larger standard deviations of the two-mode mixture, which generate a less concentrated distribution over preorder types. Thus, the number of modes alone does not determine entropy: a distribution with fewer but wider modes may be more heterogeneous than a distribution with more but narrower modes.

Overall, the simulation shows that the entropy-based heterogeneity measure responds both to the number of preorder types considered and to the evenness of their empirical frequencies. Heterogeneity is maximised when the distribution over preorder

types is balanced, and decreases when probability mass is concentrated on a smaller effective subset of types.

6 Conclusions, limitations and future research

This contribution has introduced a formally defined entropy-based measure for quantifying preference heterogeneity in group decision-making problems with complete preorders. The proposal addresses a relevant challenge in the analysis of complex ordinal preference structures: how to obtain a scalable and interpretable summary of heterogeneity without relying on pairwise comparisons, metric embeddings, or the enumeration of the full preference space. In contrast to approaches based on simplified preference models or heuristic uses of entropy, the proposed measure is directly defined on the empirical distribution of complete preorder types and is supported by a rigorous mathematical foundation.

The measure satisfies key normative properties, including continuity, anonymity, and neutrality, and is axiomatically characterised. This characterisation establishes the internal coherence of the proposal and identifies Shannon’s entropy as the unique measure satisfying the stated conditions. These conditions are grounded in principles from *Social Choice Theory* and *Decision Theory*, thereby connecting the information-theoretic formulation with well-established requirements for preference aggregation and group decision analysis.

From a computational perspective, the measure exhibits linear complexity with respect to the number of agents once the observed preorder frequencies are available. This makes it suitable for large-scale preference profiles, where distance-based or pairwise comparison procedures may become computationally demanding. The empirical analysis, based on ten real ranked-choice elections from the *PrefLib* database, illustrates the applicability of the measure to profiles with different electorate sizes, numbers of alternatives, observed preorder supports, and preference distributions. The decomposition of the relative entropy index further shows that the interpretation of $\delta_{H_R}(\mathcal{P})$ depends not only on the dispersion of the observed preorder types, but also on the rapid growth of the theoretical space of complete preorders.

The complementary comparison with selected consensus- and distance-based dissent indicators provides additional evidence on the empirical interpretation of the proposal. The results show that the entropy-based measure is broadly coherent with reference-based dissent indicators, while not being reducible to them. In particular, the proposed index captures the informational dispersion of the empirical distribution of complete preorder types, whereas reference- and distance-based measures evaluate disagreement relative to specific rankings or metric representations. The comparison therefore supports a complementary interpretation: entropy-based heterogeneity and distance-based dissent indicators describe related but distinct aspects of preference profiles.

The simulation study complements the real-data analysis by examining the behaviour of the measure under controlled distributional patterns. The results show that the measure responds coherently to changes in concentration, multimodality, skewness, and the number of distinct preorder types. These experiments provide further insight

into the sensitivity of the proposal and help clarify the interpretation of entropy-based heterogeneity when the underlying preference-generating mechanism is known.

Several limitations nevertheless remain. First, the measure summarises the frequency distribution of complete preorder types and therefore does not explicitly account for partial agreement over subsets of alternatives, such as pairwise unanimity or local agreement on specific alternatives. Second, the empirical application assumes complete and internally consistent preference information. In the real election examples, preprocessing was required to exclude incomplete or inconsistent ballots, which may affect the composition of the analysed profiles. Future research may extend the framework to partial preferences, truncated rankings, missing data, or noisy responses. One possible direction would be to redefine the underlying probability distribution over sets of admissible completions rather than over directly observed complete preorders.

A broader empirical benchmarking exercise also remains a relevant avenue for future work. Although a comparison with selected consensus- and distance-based indicators has been incorporated, a fully exhaustive assessment would require additional measures, larger collections of datasets, and a systematic analysis of computational scalability. Such an extension would be particularly useful for identifying the empirical conditions under which entropy-based, reference-based, and metric-based approaches provide convergent or divergent diagnoses of group heterogeneity.

Finally, inspired by the foundational ideas of [80] and [81], future research may further explore conceptual bridges between *Decision Theory*, *Information Theory*, and *Statistical Mechanics*. Such developments may contribute to the emergence of interdisciplinary approaches for analysing collective behaviour, preference diversity, and large-scale decision-making systems.

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Appendix A Notation and summary of properties

For ease of reference, the notation employed throughout the paper is collected in Table A1, and the main properties of the entropy-based heterogeneity preference measure, together with their normative role, are summarised in Table A2.

Table A1: Notation used throughout the paper.

Symbol	Description
$\mathbf{N} = \{1, \dots, n\}$	Set of agents, individuals, or experts.
$n = \mathbf{N} $	Number of agents ($n = \mathbf{N} > 2$).
$\mathbf{X} = \{x_1, \dots, x_k\}$	Set of alternatives.
$k = \mathbf{X} $	Number of alternatives ($k \geq 2$).
$\mathbf{W}(\mathbf{X})$	Set of complete preorders over $\mathbf{X}$.
$\phi(k)$	Number of complete preorders on k alternatives, $\phi(k) = \mathbf{W}(\mathbf{X}) $.
$\mathcal{R}_i \in \mathbf{W}(\mathbf{X})$	Preference, represented as a complete preorder, of agent i .
$x_s \mathcal{R}_i x_t$	x_s is at least as good as x_t for agent i .
$\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n)$	Preference profile of the agents in $\mathbf{N}$.
$(\mathcal{R}, \dots, \mathcal{R})$	Unanimous profile, in which all agents share the preference $\mathcal{R}$.
$\mathbf{P}(\mathbf{X})$	Set of all preference profiles over $\mathbf{X}$.
$\pi \mathcal{P}$	Profile obtained from $\mathcal{P}$ by a permutation π of the alternatives.
$\mathcal{P}^\sigma$	Profile obtained from $\mathcal{P}$ by a permutation σ of the agents.
$\mathcal{P}^I$	Subprofile of $\mathcal{P}$ restricted to the agents in $I \subseteq \mathbf{N}$, written $\mathcal{P}^i$ when $I = \{i\}$.
$\mathcal{R}_{ \mathbf{X}_A}$	Restriction of the preference $\mathcal{R}$ to $\mathbf{X}_A \subseteq \mathbf{X}$.
$\mathcal{P}_{ \mathbf{X}_A}$	Restriction of the profile $\mathcal{P}$ to $\mathbf{X}_A \subseteq \mathbf{X}$.
$\bar{\mathbf{N}}^q$	Extended agent set, including q additional agents.
$\bar{\mathcal{P}}^q$	Extension of a preference profile to the q new agents.
$\tilde{\mathbf{X}}^q$	Expanded set of alternatives, with q new options.
$\mathbf{W}(\tilde{\mathbf{X}}^q)$	Set of complete preorders over the expanded set $\tilde{\mathbf{X}}^q$.
$\tilde{\mathcal{R}}_i$	Expanded preference of agent i over $\tilde{\mathbf{X}}^q$.
$\tilde{\mathcal{P}}^q$	Expanded preference profile over $\tilde{\mathbf{X}}^q$.
$\mathbf{P}(\tilde{\mathbf{X}}^q)$	Set of all expanded preference profiles.
$\mathcal{P} \uplus \mathcal{P}$	Replication of profile $\mathcal{P}$, in which each preference appears twice.
$p(\mathcal{P}) = (p_1, \dots, p_{\phi(k)})$	Discrete probability distribution associated with $\mathcal{P}$, where p_i is the proportion of agents in $\mathcal{P}$ holding the i -th preorder.
$\#\{\mathcal{R}_j \in \mathcal{P} : \mathcal{R}_j \equiv \mathcal{R}_i\}$	Number of agents in $\mathcal{P}$ whose preference equals the i -th preorder in $\mathbf{W}(\mathbf{X})$.
$\delta : \mathbf{P}(\mathbf{X}) \rightarrow [0, \infty)$	Heterogeneity preference measure (Definition 2).
$\delta_H(\mathcal{P})$	Entropy-based heterogeneity preference measure (Definition 3).
$\delta_{H_R}(\mathcal{P})$	Relative, or normalised, entropy-based measure $\delta_{H_R}(\mathcal{P}) = \delta_H(\mathcal{P}) / \log \phi(k) \in [0, 1]$ (Definition 4).

Table A2: Main properties of the entropy-based heterogeneity preference measure and their normative role.

Property	Description	Normative role
Continuity	The measure is continuous and can be extended from $[0, 1]^{\phi(k)}$ to $\mathbb{R}^+$.	Ensures robustness: small perturbations of the profile, such as sampling variability or marginal changes in group composition, produce only small changes in the measure, thereby supporting stable comparisons across scenarios.
Non-negativity	The measure is strictly positive for non-unanimous profiles and equals zero only under unanimity.	Guarantees interpretability as a departure from homogeneity: it provides a genuine index of disagreement that vanishes exactly at consensus.
Boundedness	The maximum value is $\log \phi(k)$, achieved when all complete pre-orders are equally likely.	Provides a natural benchmark, the maximum $\log \phi(k)$, against which observed heterogeneity is assessed, and underlies the normalised index δ_{H_R} .
Grouping	If two subgroups with different unanimous preferences merge into a single unanimous group, the measure decreases by a computable quantity.	Quantifies the homogeneity gain when two subgroups adopt a common position, that is, the consensus achieved in a deliberation or negotiation round; it is also the recursivity axiom underlying the characterisation (Theorem 2).
Alternative invariance	Adding alternatives that are unanimously ranked by all agents does not change the measure.	Ensures that heterogeneity reflects the substance of preferences, not their representation: alternatives on which all agents agree do not register as additional disagreement.
Replication invariance	Duplicating the profile, that is, replicating the agents, does not affect the value of the measure.	Confers scale invariance with respect to group size: the measure captures the structure of preferences, not the absolute number of agents.
Convergence to unanimity	As more identical agents are added, the measure tends to zero, reflecting convergence to unanimity.	Ensures consistency with the consensus limit: progressively adding identical agents drives the measure to zero, matching the intuition of approaching full homogeneity.

Appendix B Proofs of properties in Subsection 4.1

Proof of property (iii). Boundedness.

Let $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{P}(\mathbf{X})$ be a preference profile and $(p_i)_{i=1}^{\phi(k)}$ its discrete probability distribution. It is enough to check that the solution of the following optimisation problem is attained at the point $\left(p_i = \frac{1}{\phi(k)}\right)_{i=1}^{\phi(k)}$:

$$\begin{aligned} \max \delta_H(\mathcal{P}) &:= - \sum_{i=1}^{\phi(k)} p_i \log p_i \\ \text{s.a.} \quad &\sum_{i=1}^{\phi(k)} p_i = 1 \\ &0 \leq p_i \leq 1, \quad \text{for all } i = 1, \dots, \phi(k). \end{aligned}$$

Therefore, the maximum value of the entropy-based heterogeneity preference measure is $\log \phi(k)$. $\square$

Proof of property (iv). Grouping.

Suppose that $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ is a preference profile and that I_1 and I_2 are two disjoint subsets of $\mathbf{N}$ with cardinalities $|I_1| = n_1$ and $|I_2| = n_2$, respectively. These two sets induce the partition $\{I_1, I_2, \mathbf{N} \setminus (I_1 \cup I_2)\}$. It is assumed that the agents' opinions on $\mathbf{X}$ are unanimous within each set, but different between the two sets. Let $\mathcal{P}^{I_1} = (\mathcal{R}_1, \dots, \mathcal{R}_1) \in \mathbf{P}(\mathbf{X})$ be the preference subprofile of the agents in I_1 , where $\mathcal{R}_1$ is their unanimous preference on $\mathbf{X}$, and let $\mathcal{P}^{I_2} = (\mathcal{R}_2, \dots, \mathcal{R}_2) \in \mathbf{P}(\mathbf{X})$ be the preference subprofile of the agents in I_2 , where $\mathcal{R}_2$ is their unanimous preference on $\mathbf{X}$, with $\mathcal{R}_1 \neq \mathcal{R}_2$. Moreover, agents in $\mathbf{N} \setminus (I_1 \cup I_2)$ have opinions different from $\mathcal{R}_1$ and $\mathcal{R}_2$.

All agents included in I_2 change their preferences on $\mathbf{X}$ in such a way that they unanimously abandon $\mathcal{R}_2$ and unanimously adopt $\mathcal{R}_1$, generating the preference profile $\overline{\mathcal{P}}$. The discrete probability distributions of the profiles are first examined:

- For $\mathcal{P}$, its discrete probability distribution is given by

$$\{p_{\alpha_i}(\mathcal{P})\}_{i=1}^m = \{p_{\alpha_1}, \dots, p_{\alpha_m}\},$$

where m represents the number of different preorders in $\mathcal{P}$. Without loss of generality, it is assumed that p_{α_1} and p_{α_2} are the terms associated with $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.⁶

⁶It should be noted that the profile dependence has been omitted.

- For $\mathcal{P}^{I_1 \cup I_2}$, its discrete probability distribution is defined as

$$\left\{ \frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}}, \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}} \right\},$$

where $p_{\alpha_1} = \frac{n_1}{n}$ and $p_{\alpha_2} = \frac{n_2}{n}$ because there are n_1 agents in $\mathbf{N}$ with $\mathcal{R}_1$ as their unanimous preference on $\mathbf{X}$ and n_2 agents in $\mathbf{N}$ with $\mathcal{R}_2$ as their unanimous preference on $\mathbf{X}$.

- For the profile $\bar{\mathcal{P}}$, the associated discrete probability distribution is

$$\{p_{\alpha_1} + p_{\alpha_2}, p_{\alpha_3}, \dots, p_{\alpha_m}\},$$

where, similarly, m represents the number of different preorders in $\mathcal{P}$.

The following identity, which is equivalent to the proposed claim, will now be proved:

$$\begin{aligned} \delta_H(\{p_{\alpha_1}, \dots, p_{\alpha_m}\}) &= (p_{\alpha_1} + p_{\alpha_2}) \delta_H\left(\left\{\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}}, \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}}\right\}\right) \\ &\quad + \delta_H(\{p_{\alpha_1} + p_{\alpha_2}, p_{\alpha_3}, \dots, p_{\alpha_m}\}). \end{aligned}$$

The proof is a simple algebraic manipulation in which the first two terms, $\{p_{\alpha_1}, p_{\alpha_2}\}$, are treated separately. Using the definition of $\delta_H(\cdot)$ and elementary properties of the logarithm, it follows that

$$\begin{aligned} \delta_H(\{p_{\alpha_1}, \dots, p_{\alpha_m}\}) &= - \sum_{i=1}^m p_{\alpha_i} \log p_{\alpha_i} \\ &= -p_{\alpha_1} \log p_{\alpha_1} - p_{\alpha_2} \log p_{\alpha_2} - \sum_{i=3}^m p_{\alpha_i} \log p_{\alpha_i} \\ &= -p_{\alpha_1} \log p_{\alpha_1} - p_{\alpha_2} \log p_{\alpha_2} \pm (p_{\alpha_1} + p_{\alpha_2}) \log (p_{\alpha_1} + p_{\alpha_2}) - \sum_{i=3}^m p_{\alpha_i} \log p_{\alpha_i} \\ &= (p_{\alpha_1} + p_{\alpha_2}) \underbrace{\left(-\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}} \log \left(\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}} \right) - \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}} \log \left(\frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}} \right) \right)}_{=\delta_H\left(\left\{\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}}, \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}}\right\}\right)} \\ &\quad + \underbrace{\left(-(p_{\alpha_1} + p_{\alpha_2}) \log (p_{\alpha_1} + p_{\alpha_2}) - \sum_{i=3}^m p_{\alpha_i} \log p_{\alpha_i} \right)}_{=\delta_H(\{p_{\alpha_1} + p_{\alpha_2}, p_{\alpha_3}, \dots, p_{\alpha_m}\})}. \end{aligned}$$

Rearranging terms, and considering the distributions of the profiles $\mathcal{P}$, $\mathcal{P}^{I_1 \cup I_2}$ and $\overline{\mathcal{P}}$ defined above, the result follows:

$$\delta_H(\overline{\mathcal{P}}) = \delta_H(\mathcal{P}) - \frac{n_1 + n_2}{n} \delta_H(\mathcal{P}^{I_1 \cup I_2}).$$

□

Proof of property (v). Alternative invariance.

Let $\mathcal{P} \in \mathbf{P}(\mathbf{X})$ be a preference profile and $(p_i(\mathcal{P}))_{i=1}^{\phi(k)}$ its discrete probability distribution. Additionally, let $\tilde{\mathcal{P}}^q \in \mathbf{P}(\tilde{\mathbf{X}}^q)$ be an expanded preference profile and $(p_i(\tilde{\mathcal{P}}^q))_{i=1}^{\phi(k+q)}$ its discrete probability distribution.

Under the assumptions of Alternative invariance, there is a bijection between the preference types observed in $\mathcal{P}$ and the corresponding expanded preference types observed in $\tilde{\mathcal{P}}^q$. Therefore, this association preserves the corresponding relative frequencies.

Consequently, the strictly positive components of the discrete probability distributions induced by $\mathcal{P}$ and $\tilde{\mathcal{P}}^q$ coincide. Hence,

$$\delta_H(\mathcal{P}) = \delta_H(\tilde{\mathcal{P}}^q).$$

□

Proof of property (vi). Replication invariance.

Let $\mathcal{P} \in \mathbf{W}(\mathbf{X})^n$ be a preference profile and $\mathcal{P} \uplus \mathcal{P} \in \mathbf{W}(\mathbf{X})^{2n}$ be a replication of $\mathcal{P}$.

The proof is straightforward, since

$$p_i(\mathcal{P} \uplus \mathcal{P}) = \frac{\#\{\mathcal{R}_j \in \mathcal{P} \uplus \mathcal{P} : \mathcal{R}_i \equiv \mathcal{R}_j\}}{2n} = \frac{2\#\{\mathcal{R}_j \in \mathcal{P} : \mathcal{R}_i \equiv \mathcal{R}_j\}}{2n} = p_i(\mathcal{P}).$$

Then,

$$\delta_H(\mathcal{P}) = \delta_H(\mathcal{P} \uplus \mathcal{P}).$$

□

Proof of property (vii). Convergence to unanimity.

Let $\mathbf{N}$ be an agent set and $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_n) \in \mathbf{W}(\mathbf{X})^n$ be a preference profile associated with $\mathbf{N}$. Suppose that q new agents are added to $\mathbf{N}$, $\tilde{\mathbf{N}}^q = \{1, \dots, n, n+1, \dots, n+q\}$, and suppose that the extension of $\mathcal{P}$ associated with $\tilde{\mathbf{N}}^q$, $\tilde{\mathcal{P}}^q = (\mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{R}_{n+1}, \dots, \mathcal{R}_{n+q}) \in \mathbf{W}(\mathbf{X})^{n+q}$, is defined such that $\mathcal{R}_n = \mathcal{R}_{n+1} = \dots = \mathcal{R}_{n+q}$. Let $\ell \in \{1, \dots, \phi(k)\}$ denote the index, in the fixed enumeration of $\mathbf{W}(\mathbf{X})$, of the unanimous preference shared by the agents $n, n+1, \dots, n+q$.

By definition,

$$p_i(\bar{\mathcal{P}}^q) = \frac{n p_i(\mathcal{P})}{n+q} \quad \text{for all } i \in \{1, \dots, \phi(k)\} \setminus \{\ell\}, \quad p_\ell(\bar{\mathcal{P}}^q) = \frac{n p_\ell(\mathcal{P}) + q}{n+q}.$$

Hence,

$$\lim_{q \rightarrow \infty} p_i(\bar{\mathcal{P}}^q) = 0 \quad \text{for all } i \neq \ell, \quad \lim_{q \rightarrow \infty} p_\ell(\bar{\mathcal{P}}^q) = 1.$$

Therefore, by continuity,

$$\lim_{q \rightarrow \infty} \delta_H(\bar{\mathcal{P}}^q) = 0.$$

□

Appendix C Proof of Subsection 4.2

The proof of this result is an adaptation of Tverberg's characterisation to the present context and is included to make the contribution self-contained. Necessity follows from the properties presented in Subsection 4.1. Sufficiency naturally falls into three steps.

First Step: The case of two different preorders. Consider a set of agents $\mathbf{N} = \{1, \dots, n\}$ who express their preferences over a set of at most two elements of $\mathbf{W}(\mathbf{X})$. The discrete probability distribution of any associated preference profile $\mathcal{P}$ is of the form

$$p(\mathcal{P}) = (0, \dots, 0, p, 0, \dots, 0, 1 - p, 0, \dots, 0) \equiv \{p, 1 - p\},$$

that is, all agents express their opinions through at most two different preorders. Let $\theta(p)$ be the one-variable function defined by

$$\theta(p) = \delta(p(\mathcal{P})) = \delta(\{p, 1 - p\}), \quad \text{with } p \in [0, 1].$$

In this step, it is first proved that $\theta(p)$ is

$$\theta(p) = -\lambda(p \log p + (1 - p) \log(1 - p)), \quad \lambda \in \mathbb{R}^+. \quad (\text{C1})$$

Since δ is a heterogeneity measure, it is non-negative and satisfies neutrality and property (i). Hence, it follows that

- $\theta(p) > 0$ if $p \neq 0$ and $\theta(p) = 0$ if $p = 0$ (the associated $\mathcal{P}$ is unanimous).
- $\theta : [0, 1] \rightarrow \mathbb{R}$ is continuous.
- $\theta(\cdot)$ is symmetric, i.e., $\theta(p) = \theta(1 - p)$.

Let $p(P) = \{p_{\alpha_1}, p_{\alpha_2}, p_{\alpha_3}\}$ now be a profile with three strictly positive components, $p_{\alpha_1}, p_{\alpha_2}, p_{\alpha_3} > 0$. This can be assumed, without loss of generality, because $n > 2$ and $k \geq 2$. Combining neutrality and the grouping property (iv), $\delta(P)$ can be written in terms of $\theta(\cdot)$ as follows:

$$\begin{aligned} \delta(\mathcal{P}) &= \delta(\{p_{\alpha_1}, p_{\alpha_2}, p_{\alpha_3}\}) \stackrel{(iv)}{=} \delta(\{p_{\alpha_1} + p_{\alpha_2}, p_{\alpha_3}\}) + (p_{\alpha_1} + p_{\alpha_2}) \delta\left(\left\{\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}}, \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}}\right\}\right) \\ &\stackrel{(\theta \text{ def.})}{=} \theta(p_{\alpha_3}) + (1 - p_{\alpha_3}) \theta\left(\frac{p_{\alpha_2}}{1 - p_{\alpha_3}}\right), \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} \delta(\mathcal{P}) &\stackrel{(neut.)}{=} \delta(\{p_{\alpha_1}, p_{\alpha_3}, p_{\alpha_2}\}) \stackrel{(iv)}{=} \delta(\{p_{\alpha_1} + p_{\alpha_3}, p_{\alpha_2}\}) + (p_{\alpha_1} + p_{\alpha_3}) \delta\left(\left\{\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_3}}, \frac{p_{\alpha_3}}{p_{\alpha_1} + p_{\alpha_3}}\right\}\right) \\ &\stackrel{(\theta \text{ def.})}{=} \theta(p_{\alpha_2}) + (1 - p_{\alpha_2}) \theta\left(\frac{p_{\alpha_3}}{1 - p_{\alpha_2}}\right). \end{aligned} \quad (\text{C3})$$

In equation (C3), $p_{\alpha_1} + p_{\alpha_2} + p_{\alpha_3} = 1$ has been used and $\theta(p) = \delta(\{p, 1 - p\}) = \theta(1 - p)$. A change of variables ($p \equiv p_{\alpha_3}$, $q \equiv p_{\alpha_2}$) yields

((C2) $\equiv$ (C3)):

$$\theta(p) + (1-p)\theta\left(\frac{q}{1-p}\right) = \theta(q) + (1-q)\theta\left(\frac{p}{1-q}\right), \quad 0 < p, q < 1. \quad (\text{C4})$$

Note that $0 < p + q < 1$, and therefore $\frac{q}{1-p}, \frac{p}{1-q} \in (0, 1)$. This functional equation is now manipulated to obtain equation (C1). Continuity of $\theta(\cdot)$ ensures the integrability of (C4) with respect to q between 0 and $(1-p)$:

$$\int_0^{1-p} \left[\theta(p) + (1-p)\theta\left(\frac{q}{1-p}\right) \right] dq = \int_0^{1-p} \left[\theta(q) + (1-q)\theta\left(\frac{p}{1-q}\right) \right] dq. \quad (\text{C5})$$

Then, each term is formally integrated, obtaining

$$\begin{aligned} \int_0^{1-p} \theta(p) dq &= (1-p)\theta(p), & \int_0^{1-p} (1-p)\theta\left(\frac{q}{1-p}\right) dq &\stackrel{t=q/(1-p)}{=} (1-p)^2 \int_0^1 \theta(t) dt. \\ \int_0^{1-p} \theta(q) dq, & & \int_0^{1-p} (1-q)\theta\left(\frac{p}{1-q}\right) dq &\stackrel{t=p/(1-q)}{=} p^2 \int_p^1 t^{-3}\theta(t) dt. \end{aligned}$$

Substituting these expressions into (C5) gives

$$(1-p)\theta(p) + (1-p)^2 \int_0^1 \theta(t) dt = \int_0^{1-p} \theta(t) dt + p^2 \int_p^1 t^{-3}\theta(t) dt. \quad (\text{C6})$$

Note that the first term in equation (C6) is differentiable because the continuity of $\theta(\cdot)$ implies the differentiability of the other terms. Upon differentiation, equation (C6) gives

$$-\theta(p) + (1-p)\theta'(p) - 2(1-p) \int_0^1 \theta(t) dt = -\theta(1-p) + 2p \int_p^1 t^{-3}\theta(t) dt - \frac{\theta(p)}{p}. \quad (\text{C7})$$

Given that $\theta(\cdot)$ is symmetric, i.e., $\theta(p) = \theta(1-p)$, reorganising terms yields

$$(1-p)\theta'(p) = 2(1-p) \int_0^1 \theta(t) dt + 2p \int_p^1 t^{-3}\theta(t) dt - \frac{\theta(p)}{p}. \quad (\text{C8})$$

The previous argument again ensures the differentiability of $(1-p)\theta'(p)$. Differentiating with respect to p gives

$$-\theta'(p) + (1-p)\theta''(p) = -2 \int_0^1 \theta(t) dt + 2 \int_p^1 t^{-3}\theta(t) dt - 2p \frac{\theta(p)}{p^3} - \frac{\theta'(p)}{p} + \frac{\theta(p)}{p^2} \quad (\text{C9})$$

$$\Rightarrow \quad (1-p)\theta''(p) = -2 \int_0^1 \theta(t) dt + 2 \int_p^1 t^{-3} \theta(t) dt - \frac{\theta(p)}{p^2} - \frac{\theta'(p)}{p} + \theta'(p). \quad (\text{C10})$$

In order to eliminate the term $\int_p^1 t^{-3} \theta(t) dt$, the last equation is multiplied by p , and the result is subtracted from (C8), obtaining

$$-(1-p)p\theta''(p) = 2 \int_0^1 \theta(t) dt \quad \Rightarrow \quad \theta''(p) = -\frac{2}{p(1-p)} \int_0^1 \theta(t) dt.$$

Then, this differential equation can be integrated by quadratures⁷, to obtain

$$\theta(p) = A + Bp - 2[p \log p + (1-p) \log(1-p)] \int_0^1 \theta(t) dt.$$

On the one hand, $\theta(0) = 0$ implies $A = 0$ and, by symmetry of $\theta(\cdot)$, it follows that $B = 0$. On the other hand, it follows that $\theta(\cdot)$ is determined up to a multiplicative constant:

$$\theta(p) = -\lambda [p \log p + (1-p) \log(1-p)],$$

where $\lambda > 0$ because $\theta(\cdot)$ is non-negative. This concludes the first step.

Second Step: General case. Let $\mathcal{P}$ be a preference profile and let its discrete probability distribution be

$$p(\mathcal{P}) = (p_1, \dots, p_{\phi(k)}) \equiv \{p_{\alpha_1}, \dots, p_{\alpha_m}\},$$

where m denotes the number of different preorders in $\mathcal{P}$. In this step, it is proved that

$$\delta(\mathcal{P}) = \delta(\{p_{\alpha_1}, \dots, p_{\alpha_m}\}) = -\lambda \sum_{i=1}^m p_{\alpha_i} \log p_{\alpha_i}.$$

The case $m = 2$ was proved in the first step:

$$\delta(\{p_{\alpha_1}, p_{\alpha_2}\}) = \theta(p_{\alpha_1}) = -\lambda [p_{\alpha_1} \log p_{\alpha_1} + (1-p_{\alpha_1}) \log(1-p_{\alpha_1})] = -\lambda \sum_{i=1}^2 p_{\alpha_i} \log p_{\alpha_i}.$$

By induction, assume the result for $m-1$. Finally, for the general case, combining the grouping property (iv), the induction hypothesis (HI), and the previous step gives

$$\delta(\{p_{\alpha_1}, \dots, p_{\alpha_m}\}) \stackrel{(iv)}{=} (p_{\alpha_1} + p_{\alpha_2}) \delta\left(\left\{\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}}, \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}}\right\}\right) + \delta(\{p_{\alpha_1} + p_{\alpha_2}, p_{\alpha_3}, \dots, p_{\alpha_m}\})$$

⁷ $\int \int \left(\frac{1}{p(1-p)}\right) dt dp = \int (b + \log t - \log(1-t)) dp = A + Bp - 2(p \log p + (1-p) \log(1-p)).$

$$\begin{aligned}
& \stackrel{(HI)}{=} -(p_{\alpha_1} + p_{\alpha_2})\lambda \left[\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}} \log \left(\frac{p_{\alpha_1}}{p_{\alpha_1} + p_{\alpha_2}} \right) + \frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}} \log \left(\frac{p_{\alpha_2}}{p_{\alpha_1} + p_{\alpha_2}} \right) \right] \\
& \quad - (p_{\alpha_1} + p_{\alpha_2})\lambda \log(p_{\alpha_1} + p_{\alpha_2}) - \lambda \sum_{i=3}^m p_{\alpha_i} \log p_{\alpha_i} \\
& = -\lambda \sum_{i=1}^m p_{\alpha_i} \log p_{\alpha_i} = -\lambda \sum_{i=1}^{\phi(k)} p_i \log p_i.
\end{aligned}$$

This concludes the second step of the proof.

Third Step: Fixing the constant. In order to fix the value of λ , the boundedness property (iii) is invoked. Let $\mathcal{P}$ be a preference profile $\mathcal{P} = (\mathcal{R}_1, \dots, \mathcal{R}_{\phi(k)}) \in \mathbf{W}(\mathbf{X})^{\phi(k)}$ with $\mathcal{R}_i \neq \mathcal{R}_j$ for all $i \neq j$ and with discrete probability distribution $p(\mathcal{P}) = (p_{\alpha_1}, \dots, p_{\alpha_{\phi(k)}}) = \underbrace{\left\{ \frac{1}{\phi(k)}, \dots, \frac{1}{\phi(k)} \right\}}_{\phi(k)}$. Let property (iii) hold. Then:

$$\log \phi(k) = \delta(\mathcal{P}) = -\lambda \sum_{i=1}^{\phi(k)} \frac{1}{\phi(k)} \log \left(\frac{1}{\phi(k)} \right) \quad \Rightarrow \quad \lambda = 1.$$

Therefore, properties (i), (iii), and (iv) characterise the entropy-based heterogeneity preference measure, $\delta_H(\cdot)$, and the proof is complete. $\square$

Remark 7 Concerning the mutual independence of the axioms in Theorem 2, a comprehensive proof for all hypotheses simultaneously falls outside the scope of this paper because of its mathematical complexity. Nevertheless, their independence can be briefly justified as follows. The boundedness axiom merely fixes the function's parameter determined by the other properties. Symmetry, or neutrality, is independent, since the function $\theta(p) = p$ satisfies both continuity and grouping but is not symmetric. Similarly, the grouping axiom is independent, since functions such as $\theta(p) = p(1 - p)$ are symmetric and continuous yet fail to satisfy this property. Finally, proving the independence of continuity is significantly more arduous; however, [82] established the existence of non-continuous solutions that still satisfy the grouping property.

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